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Breaking Index Study on Weighted Laplace Equation

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***Abstract*— This study serves as an extension of prior research focusing on weighting coefficients within the context of weighted Taylor series. The primary objective is to determine the weighting coefficients' values in the weighted Taylor series for the purpose of modeling water waves based on velocity potential. Utilizing the weighted Taylor series, we derive both the weighted continuity equation and the weighted Laplace equation. The latter is addressed using the variable separation method over a sloping bottom, leading to the formulation of the velocity potential equation, wave constant equations, and energy conservation equations. Within the wave constant equations, a breaking equation is incorporated. Leveraging both the breaking equation and the energy conservation equations, breaking indexes equations are formulated. These equations encompass breaker length, breaker depth, and breaker height indexes, with weighting coefficients prominently featured. Calibrating the results of the breaking indexes equations against findings from earlier studies provides suitable values for the weighting coefficients. Additionally, this research introduces a shoaling-breaking model and a refraction-diffraction model to explore the phenomena of shoaling-breaking within the solution of the weighted Laplace equation.*

I. INTRODUCTION

Hydrodynamic equations are conventionally expressed through Taylor series, typically truncated to first order, under the assumption that higher-order terms, such as second order and beyond, become negligible at very small intervals. While not inherently incorrect, this truncation approach sacrifices the representation of the function's characteristics encapsulated in the higher-order terms, leading to an imprecise approximation.

To address this limitation, Hutahaean (2021,2022,2023a) proposes the utilization of weighted Taylor series. This involves truncating the Taylor series to the first order, while compensating for the omission of higher-order contributions through the incorporation of weighting coefficients. In Hutahaean's work (2023a), a more systematic application of Taylor series truncation is introduced, employing the Central Difference Method. This method allows for the systematic removal of even higher-order differential terms, ensuring their disappearance without merely being eliminated or overlooked. Furthermore, the study explores the extraction of 3rd order odd differential term contributions, with the additional inclusion of 5th order terms.

The use of the weighted Taylor series is characterized by the absence of truncation errors. Consequently, when applied to the continuity equation, the weighted Taylor series formulation minimizes or eliminates truncation errors. Similarly, the weighted Laplace equation, derived from the weighted continuity equation, maintains this heightened accuracy. Moreover, equations pertaining to various wave mechanics, formulated using the weighted Laplace equations, incorporate weighting coefficients, thereby enhancing their accuracy and reliability.

II. WEIGHTED TAYLOR SERIES

Taylor series (Arden, Bruce W. and Astill Kenneth N. ,1970) for a function $f = f(x,t)$ where x is the horizontal axis and t is time

$$f(x+\delta x, t+\delta t) = f(x,t) + \delta t \frac{\partial f}{\partial t} + \delta x \frac{\partial f}{\partial x} + \frac{\delta t^2}{2!}\frac{\partial^2 f}{\partial t^2} + \delta t \delta x \frac{\partial^2 f}{\partial t \partial x} + \frac{\delta x^2}{2!}\frac{\partial^2 f}{\partial x^2} + \cdots \quad \ldots\ldots\ldots(1)$$

This equation can be expressed as,

$$f(x+\delta x, t+\delta t) = f(x,t) + \left(1 + \frac{\delta t}{2!}\frac{\partial}{\partial t} + \delta x \frac{\partial}{\partial x} + \cdots\right)\delta t \frac{\partial f}{\partial t} + \left(1 + \frac{\delta x}{2!}\frac{\partial}{\partial x} + ..\right)\delta x \frac{\partial f}{\partial x}$$

In this paper, x represents the horizontal axis, z represents the vertical axis, while t represents time.

The complete Taylor series, there are contributions from high-order differential terms in the first-order differential term. The contributions of these high-order differential terms can be represented by a coefficient, allowing for the elimination of terms with high differentials.

$$f(x+\delta x, t+\delta t) = f(x,t) + \gamma_{t,2}\delta t \frac{\partial f}{\partial t} + \gamma_x \delta x \frac{\partial f}{\partial x} \quad \ldots\ldots..(2)$$

$\gamma_{t,2}$ and γ_x are referred to as weighting coefficients, while (2) is called the weighted Taylor series. The weighted Taylor series for the function $f(x,z,t)$ is as follows:

$$f(x+\delta x, z+\delta z, t+\delta t) = f(x,z,t) + \gamma_{t,3}\delta t \frac{\partial f}{\partial t} + \gamma_x \delta x \frac{\partial f}{\partial x} + \gamma_z \delta z \frac{\partial f}{\partial x} \quad \ldots\ldots(3)$$

There is no difference in the values of γ_x in the function $f(x,t)$ compared to γ_x in the function $f(x,z,t)$. The baseline values for these weighting coefficients are $\gamma_{t,2} = 2$, $\gamma_{t,3} = 3$, $\gamma_x = 1$ and $\gamma_z = 1$. More precise values of the weighting coefficients are presented in Table (1). The calculation method can be found in Hutahaean (2023a), where the computation of the weighting coefficients in Table (1) is achieved with greater accuracy compared to Hutahaean (2023a). In these weighting coefficients, all even-order differential terms in the Taylor series are represented, while the absorbed odd-order high differential terms are of order 3 and 5. Higher-order differentials are truncated with the reduction of intervals, δt, δx and δz. The values of the weighting coefficients are presented in Table (1).

Table 1 Weighting coefficient values

ε	$\gamma_{t,2}$	$\gamma_{t,3}$	γ_x	γ_z
0.03	1.99812	3.05087	0.98899	1.11999
0.032	1.99786	3.05877	0.98746	1.139
0.034	1.99758	3.06738	0.98583	1.15986
0.036	1.99728	3.07674	0.9841	1.1827
0.038	1.99695	3.08689	0.98227	1.20765
0.04	1.99662	3.09786	0.98034	1.23485
0.042	1.99626	3.1097	0.97831	1.26448
0.044	1.99588	3.12244	0.97617	1.29669
0.046	1.99548	3.13614	0.97393	1.3317

ε is referred to as the optimization coefficient, where a larger ε corresponds to a larger interval size and consequently, larger values for the absorbed third and fifth-order differential terms in the weighting coefficients. This research aims to obtain appropriate values for the weighting coefficients expressed in terms of ε. The appropriateness is assessed concerning the breaking parameter generated by the breaking index equations.

III. WEIGHTED LAPLACE EQUATION

The Laplace equation is formulated by employing the continuity equation, substituting the properties of the velocity potential into the continuity equation. The formulation of the weighted continuity equation is carried out using a well-known method, namely by applying the principle of mass conservation to a control volume where fluid inflow-outflow occurs,

3.1. Weighted Continuity Equation

By employing the weighted Taylor series, a weighted continuity equation can be obtained.

$$\gamma_x \frac{\partial u}{\partial x} + \gamma_z \frac{\partial w}{\partial z} = 0 \quad \ldots\ldots(4)$$

u is the horizontal water particle velocity, and w is the vertical water particle velocity. This equation represents the weighted continuity equation, where γ_x and γ_z are weighting coefficients. By using the weighted Taylor series, there is no longer truncation error or at least the truncation error has been greatly minimized

Mathematically, equation (4) can be written as,

$$\frac{\partial u}{\partial x}+\frac{\gamma_z}{\gamma_x}\frac{\partial w}{\partial z}=0$$

Or,

$$\frac{\gamma_x}{\gamma_z}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$$

Both of these writings are somewhat inaccurate and will result in different solutions because each coefficient has a specific function. It can be said that γ_x is associated with $\frac{\partial u}{\partial x}$ and γ_z is associated with $\frac{\partial w}{\partial z}$.

3.2. Weighted Laplace Equation

In fluid flow, there is a scalar quantity called velocity potential, denoted as,

$u=-\frac{\partial \phi}{\partial x}$ and $w=-\frac{\partial \phi}{\partial z}$

Substituting the velocity potential property into (4), we obtain the weighted Laplace equation,

$$\gamma_x\frac{\partial^2\phi}{\partial x^2}+\gamma_z\frac{\partial^2\phi}{\partial z^2}=0 \qquad \text{.......(5)}$$

This Laplace equation differs from the commonly used Laplace equation in water wave modeling, where there are actually weighting coefficients with $\gamma_x=\gamma_z=1$.

IV. SOLUTION TO THE WEIGHTED LAPLACE EQUATION

4.1. Solution using the Separation of Variables Method

The solution to (5) is carried out using the separation of variables method. In the variable separation method, it is assumed that the velocity potential is the product of three functions (Dean (1991)), namely

$$\phi(x,z,t)=X(x)Z(z)T(t) \qquad \text{....(6)}$$

Here, $X(x)$: is a function of x only, $Z(z)$ is a function of z only, and $T(t)$ is a function of t only. Substituting into (5) and dividing the equation by (6), the following is obtained:

$$\frac{\gamma_x}{X(x)}\frac{\partial^2 X}{\partial x^2}+\frac{\gamma_z}{Z(z)}\frac{\partial^2 Z}{\partial z^2}=0$$

This equation is fulfilled if

$$\frac{\gamma_x}{X(x)}\frac{\partial^2 X}{\partial x^2}=-k^2$$

Defined as $k_x=\frac{k}{\sqrt{\gamma_x}}$, hence

$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2}=-k_x^2 \qquad \text{....(7)}$$

Using the same method,

$$\frac{1}{Z(z)}\frac{\partial^2 Z}{\partial z^2}=k_z^2 \qquad \text{......(8)}$$

Where $k_z=\frac{k}{\sqrt{\gamma_z}}$.

k is the wave number where the wavelength is $L=\frac{2\pi}{k}$. Hence, there are two wavelengths: the horizontal wavelength, $L_x=\frac{2\pi}{k_x}$, and the vertical wavelength, $L_z=\frac{2\pi}{k_z}$, with different lengths.

Equation (7), is offered for a solution

$X(x)=A\cos k_x x+B\sin k_x x$

Equation (8) has a solution of

$Z(z)=Ce^{k_z z}+De^{-k_z z}$

Then, an assumption is made that the velocity potential is periodic with respect to time; hence, $T(t)=\sin\sigma t$

$\sigma=\frac{2\pi}{T}$ s the angular frequency, where T is the wave period. Substituting $X(x)$, $Z(z)$ and $T(t)$ ke (6),

$$\phi(x,z,t)=(A\cos k_x x+B\sin k_x x)\left(Ce^{k_z z}+De^{-k_z z}\right)\sin\sigma t \qquad \text{....(9)}$$

The constants A, B, C, and D in equation (9) still need their specific forms to be determined.

4.2. Working on the Kinematic Bottom Boundary Condition

To obtain equations for the constants in the solution, the Kinematic Bottom Boundary Condition is applied at characteristic points where $\cos k_x x=\sin k_x x$. At these characteristic points, the velocity potential equation becomes,

$$\phi(x,z,t)=(A+B)\cos k_x x\left(Ce^{k_z z}+De^{-k_z z}\right)\sin\sigma t$$

Kinematic bottom boundary condition is,

$$w_{-h}=-u_{-h}\frac{dh}{dx}$$

w_{-h} is the vertical water particle velocity at the sea bed at $z=-h$.

u_{-h} is the horizontal water particle velocity at the sea bed at $z=-h$.

h is the water depth relative to the still water level, $\frac{dh}{dx}$ is the bottom slope, which has a negative value for waves moving from deeper to shallower waters. Utilizing the properties of velocity potential,

$$w(x,z,t)=-\frac{\partial \phi}{\partial z}=-(A+B)k_z\cos k_x x\left(Ce^{k_z z}-De^{-k_z z}\right)\sin\sigma t$$

$$w_{-h} = -(A+B)k_z \cos k_x x \left(Ce^{-k_z h} - De^{k_z h}\right)\sin\sigma t$$

$$u(x,z,t) = -\frac{\partial\phi}{\partial x} = (A+B)k_x \sin kx \left(Ce^{k_z z} + De^{-k_z z}\right)\sin\sigma t$$

$$u_{-h} = (A+B)k_x \sin k_x x \left(Ce^{-k_z h} + De^{k_z h}\right)\sin\sigma t$$

The substitution of the Equations w_{-h} and u_{-h} to Kinematic Bottom Boundary Condition was performed at characteristic points as well as at $(A+B)$ and and when sin(σt) is not equal to zero, yields the equation

$$k_z\left(Ce^{-k_z h} - De^{k_z h}\right) = k_x\left(Ce^{-k_z h} + De^{k_z h}\right)\frac{dh}{dx}$$

$$\left(k_z - k_x\frac{dh}{dx}\right)Ce^{-k_z h} = \left(k_z + k_x\frac{dh}{dx}\right)De^{k_z h}$$

Considering $k_x = \frac{k}{\sqrt{\gamma_x}}$ dan $k_z = \frac{k}{\sqrt{\gamma_z}}$

$$\left(\frac{1}{\sqrt{\gamma_z}} - \frac{1}{\sqrt{\gamma_x}}\frac{dh}{dx}\right)Ce^{-k_z h} = \left(\frac{1}{\sqrt{\gamma_z}} + \frac{1}{\sqrt{\gamma_x}}\frac{dh}{dx}\right)De^{k_z h}$$

$$C = De^{2k_z h}\frac{\frac{1}{\sqrt{\gamma_z}} + \frac{1}{\sqrt{\gamma_x}}\frac{dh}{dx}}{\frac{1}{\sqrt{\gamma_z}} - \frac{1}{\sqrt{\gamma_x}}\frac{dh}{dx}}$$

Defined as,

$$\alpha = \frac{\frac{1}{\sqrt{\gamma_z}} + \frac{1}{\sqrt{\gamma_x}}\frac{dh}{dx}}{\frac{1}{\sqrt{\gamma_z}} - \frac{1}{\sqrt{\gamma_x}}\frac{dh}{dx}}$$

Hence

$$C = De^{2k_z h}\alpha$$

Substituting to (7)

$$\phi(x,z,t) = (A+B)\cos k_x x \left(De^{2k_z h}\alpha e^{k_z z} + De^{-k_z z}\right)\sin\sigma t$$

$$\phi(x,z,t) = (A+B)De^{k_z h}\cos k_x x \left(\alpha e^{k_z(z+h)} + e^{-k_z(z+h)}\right)\sin\sigma t$$

Defined as,

$$\beta(z) = \frac{\alpha e^{k_z(z+h)} + e^{-k_z(z+h)}}{2}$$

$$\beta_1(z) = \frac{\alpha e^{k_z(z+h)} - e^{-k_z(z+h)}}{2}$$

Where on $\alpha = 1$,

$$\beta(z) = \cosh k_z(h+z);\ \beta_1(z) = \sinh k_z(h+z)$$

$$\phi(x,z,t) = 2(A+B)De^{k_z h}\beta(z)\cos kx \sin\sigma t$$

Defined as $A = 2A$ dan $B = 2B$, hence

$$\phi(x,z,t) = (A+B)De^{k_z h}\beta(z)\cos k_x x \sin\sigma t$$

Hutahaean (2022) shows that $A = B$

$$\phi(x,z,t) = 2\,ADe^{k_z h}\beta(z)\cos k_x x \sin\sigma t$$

Defined as $G = ADe^{k_z h}$

$$\phi(x,z,t) = 2G\beta(z)\cos k_x x \sin\sigma t$$

The full equation,

$$\phi(x,z,t) = G\beta(z)\cos k_x x \sin\sigma t + G\beta(z)\sin k_x x \sin\sigma t \qquad \text{.....(10)}$$

At the characteristic point where $\cos k_x x = \sin k_x x$, velocity potential equation becomes,

$$\phi(x,z,t) = 2G\beta(z)\cos k_x x \sin\sigma t \qquad \text{......(11)}$$

V. EQUATION FOR *G*

The equation for G is formulated by integrating the Kinematic Free Surface Boundary Condition with respect to time t and is performed at characteristic points. The form of the Kinematic Free Surface Boundary Condition uses a weighted Taylor Series for two variables $f = f(x,t)$, in this case, where the function f is the water elevation $\eta = \eta(x,t)$.

$$\eta(x+\delta x, t+\delta t) = \eta(x,t) + \gamma_{t,2}\delta t\frac{\partial\eta}{\partial t} + \gamma_x\delta x\frac{\partial\eta}{\partial x}$$

$$\frac{\eta(x+\delta x, t+\delta t) - \eta(x,t)}{\delta t} = \gamma_{t,2}\frac{\partial\eta}{\partial t} + \gamma_x\frac{\delta x}{\delta t}\frac{\partial\eta}{\partial x}$$

With δt and δx that are small, there obtained,

$$\frac{D\eta}{dt} = \gamma_{t,2}\frac{\partial\eta}{\partial t} + \gamma_x u_\eta\frac{\partial\eta}{\partial x}$$

$\frac{D\eta}{dt}$ is the total velocity of vertical surface water particle movement is denoted as w_η, while u_η represents the horizontal surface water particle velocity. Therefore, the equation for the Kinematic Free Surface Boundary Condition is.

$$w_\eta = \gamma_{t,2}\frac{\partial\eta}{\partial t} + \gamma_x u_\eta\frac{\partial\eta}{\partial x}$$

This equation is expressed as the water surface equation as follows,

$$\gamma_{t,2}\frac{\partial\eta}{\partial t} = w_\eta - \gamma_x u_\eta\frac{\partial\eta}{\partial x}$$

w_η and u_η are substituted by velocity potential (11), using the velocity potentials properties ($u = -\frac{\partial\phi}{\partial x}$, $w = -\frac{\partial\phi}{\partial z}$), and by considering that $k_x = \frac{k}{\sqrt{\gamma_x}}$ dan $k_z = \frac{k}{\sqrt{\gamma_z}}$,

$$\gamma_{t,2}\frac{\partial \eta}{\partial t} = -2Gk\left(\frac{1}{\sqrt{\gamma_z}}\beta_1(\eta)+\sqrt{\gamma_x}\beta(\eta)\frac{\partial \eta}{\partial x}\right) \cos k_x x \sin(\sigma t) \quad \ldots\ldots(12)$$

For the periodical function,

$$2Gk\left(\frac{1}{\sqrt{\gamma_z}}\beta_1(\eta)+\sqrt{\gamma_x}\beta(\eta)\frac{\partial \eta}{\partial x}\right) = constant$$

Integration (12) can be resolved by integrating $sin(\sigma t)$.

$$\eta(x,t) = \frac{2Gk}{\gamma_{t,2}\sigma}\left(\frac{1}{\sqrt{\gamma_z}}\beta_1(\eta)+\sqrt{\gamma_x}\beta(\eta)\frac{\partial \eta}{\partial x}\right) \cos k_x x \cos(\sigma t)$$

For the periodical function,

$$A = \frac{2Gk}{\gamma_{t,2}\sigma}\left(\frac{1}{\sqrt{\gamma_z}}\beta_1(\eta)+\sqrt{\gamma_x}\beta(\eta)\frac{\partial \eta}{\partial x}\right) \quad \ldots..(13)$$

Where A is wave amplitude. Hence, water surface elevation equation is presented as,

$$\eta(x,t) = A \cos k_x x \cos(\sigma t) \quad \ldots..(14)$$

$$\frac{\partial \eta}{\partial x} = -k_x A \sin k_x x \cos \sigma t$$

At the characteristic point, $cos k_x x = sin k_x x$ dan $cos(\sigma t) = sin(\sigma t)$

$$\frac{\partial \eta}{\partial x} = -\frac{k_x A}{2}$$

Substituted to (13), obtaining a wave amplitude function equation as follows.

$$A = \frac{2Gk}{\gamma_{t,2}\sigma}\left(\frac{1}{\sqrt{\gamma_z}}\beta_1\left(\frac{A}{2}\right)-\sqrt{\gamma_x}\beta\left(\frac{A}{2}\right)\frac{k_x A}{2}\right)$$

Considering $k_x = \frac{k}{\sqrt{\gamma_x}}$ and taking out $\beta\left(\frac{A}{2}\right)$ of the bracket,

$$A = \frac{2Gk\beta\left(\frac{A}{2}\right)}{\gamma_{t,2}\sigma}\left(\frac{1}{\sqrt{\gamma_z}}\frac{\beta_1\left(\frac{A}{2}\right)}{\beta\left(\frac{A}{2}\right)}-\frac{kA}{2}\right)$$

Based on the conservation law of wave number discussed in section (6), hence $\beta\left(\frac{A}{2}\right)$ and $\beta_1\left(\frac{A}{2}\right)$ are constant, where

$$\frac{\beta_1\left(\frac{A}{2}\right)}{\beta\left(\frac{A}{2}\right)} = constant$$

This constant value applies to all bodies of water, including both deep water and shallow water. In deep water where the bottom slope no longer has an effect, hence

$$\frac{\beta_1\left(\frac{A}{2}\right)}{\beta\left(\frac{A}{2}\right)} = \tan k\left(h+\frac{A}{2}\right) \approx 1$$

This condition is achieved when,

$$k\left(h+\frac{A}{2}\right) = \theta\pi$$

Where $\tanh\theta\pi \approx 1$, θ is referred to as the deep water coefficient. The equation for the wave amplitude function becomes:

$$A = \frac{2Gk\cosh\theta\pi}{\gamma_{t,2}\sigma}\left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}-\frac{kA}{2}\right) \quad \ldots..(15)$$

This equation can be expressed as Equation for G,

$$G = \frac{\sigma\gamma_{t,2}A}{2k\left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}-\frac{kA}{2}\right)\cosh\theta\pi} \quad \ldots\ldots(16)$$

VI. EQUATIONS OF CONSERVATION

In its movement towards shallower waters, waves will undergo changes in its constants, namely G, k and A. The governing equations controlling these changes are the conservation equation of wave number and the conservation equation of energy.

a. Conservation Equation of Wave Number

In the method of separating variables, it is stated that $Z(z)$ is a function of z only. For the velocity potential equation, $Z(z)$is expressed as:

$$Z(z) = \beta(z)$$

As only function z, hence,

$$\frac{\partial Z(z)}{\partial x} = \beta_1(z)\frac{\partial k_z(h+z)}{\partial x} = 0$$

Hence

$$\frac{\partial k_z(h+z)}{\partial x} = 0$$

Substituting $k_z = \frac{k}{\sqrt{\gamma_z}}$ and according to the formulation of G equation, therefore $z = \frac{A}{2}$ is used.

$$\frac{\partial k\left(h+\frac{A}{2}\right)}{\partial x} = 0 \quad \ldots\ldots(17)$$

This equation is the conservation equation of wave number. This equation shows that,,

$$k\left(h+\frac{A}{2}\right) = constant$$

The constant value is found in deep water, i.e.,

$$\tanh k\left(h+\frac{A}{2}\right) \approx 1$$

Where $\tanh\theta\pi \approx 1$ and θ is the deep water coefficient, therefore,

$$k\left(h+\frac{A}{2}\right) = \theta\pi \quad \ldots.(18)$$

This equation is valid for all water depths, including deep water and shallow water.

b. Conservation Equation of Energy

Substituting (10) into (5) using the properties of velocity potential, and working on the obtained equation with point characteristic properties, we get

$$\frac{\partial^2 G}{\partial x^2} = 0 \qquad \ldots(19)$$

Substituting (11) into (5) using the properties of velocity potential and considering (19) at characteristic points, we obtain

$$G\frac{\partial k}{\partial x} + 2k\frac{\partial G}{\partial x} = 0 \qquad \ldots.(20)$$

This is an energy conservation equation.

VII. DEEP WATER WAVE LENGTH

Hutahaean (2022) formulates the relationship between wavelength and wave height in deep water. By employing weighting coefficients in this study, the relationship between wavelength and wave height in deep water is,

$$L_0 = \frac{\pi\left(\gamma_{t,2} + \frac{\gamma_z \gamma_{t,3}}{2}\right) H_0}{\gamma_{t,3} tanh\theta\pi} \qquad \ldots.(21)$$

$$L_{0,x} = L\sqrt{\gamma_x}$$

$$L_{0,z} = L\sqrt{\gamma_z}$$

Example calculation results of the wavelength for a wave amplitude $A = 1.2$ m, with various values of

weighting coefficients, are presented in Table (2).

Table.2 Wave length at wave amplitude of 1.2 m.

ε	L_x (m)	L_z (m)	$\frac{H}{L_x}$
0.03	9.11	9.695	0.263
0.032	9.161	9.839	0.262
0.034	9.217	9.997	0.26
0.036	9.279	10.172	0.259
0.038	9.347	10.363	0.257
0.04	9.421	10.573	0.255
0.042	9.503	10.803	0.253
0.044	9.592	11.055	0.25
0.046	9.689	11.33	0.248

In Table (2), the ε values represent the weighting coefficients, with the corresponding values available in Table (1).

Table (2) includes two wavelengths: the horizontal wavelength L_x and the vertical wavelength L_z. These wavelengths exhibit a slight difference, although both are present. It is noteworthy that a larger ε corresponds to a longer wavelength, resulting in a smaller wave steepness.

Toffoli, A., Babanin, A., Onaroto, M., and Wased, T. (2010), determined that the critical wave steepness is 0.170, with a recommended upper limit of 0.200. To align with Toffoli et al.'s criteria and achieve a wave steepness closer to their recommendations, it is advisable to utilize a larger ε value.

VIII. WAVE TRANSFORMATION MODEL

This section demonstrates that the potential velocity encompasses the shoaling-breaking phenomenon. Notably, this phenomenon persists irrespective of whether the original Laplace equation (as proposed by Hutahaean (2023b)) or the weighting coefficients $\gamma_x = \gamma_z = 1$ are employed. The focus of this section is solely on illustrating that the shoaling-breaking phenomenon remains unaffected by the weighting coefficients in the Laplace equation.

Within this context, the weighting coefficient plays a crucial role in influencing the breaking characteristics. Specifically, it impacts key parameters such as breaker height H_b, breaker depth h_b and breaker length L_b.

8.1. Shoaling Breaking Modeling

The shoaling and breaking models are established based on conservation equations (17) and (20), incorporating equations governing the wave amplitude function (15) and the function G (16). The detailed formulation process is elucidated in Hutahaean (2023b). The equations governing the changes in wave constants k, A, and G as a wave travels from point x to $x + \delta x$ are essential components of this approach.

$$\frac{dk}{dx} = -\frac{4k}{(4h+3A)}\frac{dh}{dx} \qquad \ldots\ldots(22)$$

$$\frac{\partial A}{\partial x} = \frac{G}{\sigma\gamma_{t,2}}\frac{\partial k}{\partial x}\left(\frac{\tan\theta\pi}{\sqrt{\gamma_z}} - \frac{kA}{2}\right)\cosh(\theta\pi) \qquad \ldots\ldots(23)$$

Furthermore,

$$k_{x+\delta x} = k_x + \delta x\frac{\partial k}{\partial x}$$

$$A_{x+\delta x} = A_x + \delta x\frac{\partial A}{\partial x}$$

$$G_{x+\delta x} = e^{\ln G_x - \frac{1}{2}(\ln k_{x+\delta x} - \ln k_x)}$$

To demonstrate the outcomes of the shoaling-breaking model, an analysis was conducted in a coastal zone characterized by a bottom slope of $\frac{dh}{dx} = -0.02$. The study utilized waves with a period of 8.0 seconds and a deep water wave amplitude of $A_0 = 1.2$ meters. The deep water coefficient $\theta = 1.60$, and a weighting coefficient of $\varepsilon = 0.042$ was applied, with the specific values found in Table (1). The deep water wave number, k_0, was calculated using equation (21), and the deep water wave height, $h_0 = \frac{\theta \pi}{k_0} - \frac{A_0}{2}$. Additionally, the deep water wave constant G_0 was calculated using equation (16). The results of the shoaling-breaking model are visually presented in Figure (1), offering a comprehensive illustration of the dynamic behavior in the coastal zone under the specified conditions.

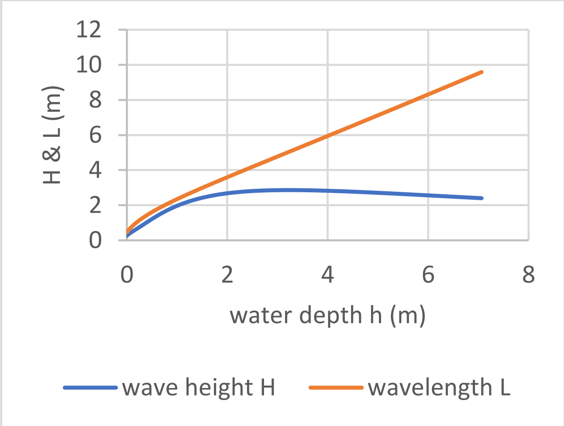

Fig.1: The outcome of shoaling-breaking modeling

In Fig(1), it can be observed that the model is able to simulate shoaling and breaking phenomena effectively. Details regarding breaking characteristics, including breaker height H_b, breaker depth h_b and breaker lengthL_b, are discussed in another section.

8.2 Refraction-Diffraction Analysis

Utilizing the shoaling-breaking equations, the formulation of refraction-diffraction equations is established, as detailed in Hutahaean (2023b). The application of the refraction-diffraction model is demonstrated on bathymetry featuring a headland configuration (Fig. (2)), considering waves with a period of 8 seconds and a deep water wave amplitude of 1.20 meters. The resulting 2-D contour image depicting wave height from the refraction-diffraction model is presented in Fig. (3), while Fig. (4) illustrates the 3-D wave height contour.

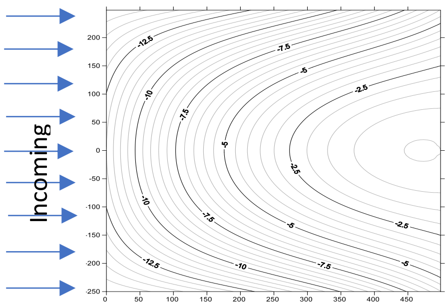

Fig.2: Batimetri tanjong contour

The model implementation involves a deep water coefficient of $\theta = 1.60$ and an optimization coefficient of $\varepsilon = 0.042$, providing insights into wave behavior in the presence of bathymetric features and headlands.

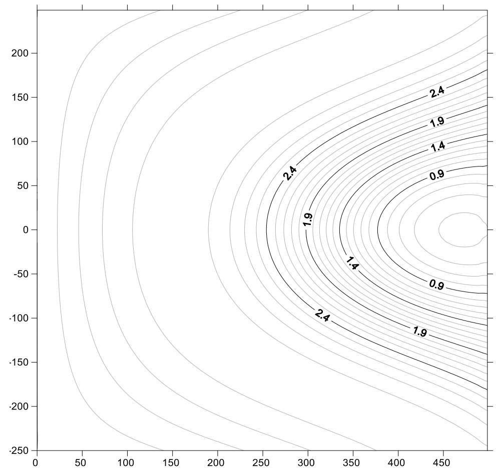

Fig.3: Wave height 2-D Contour

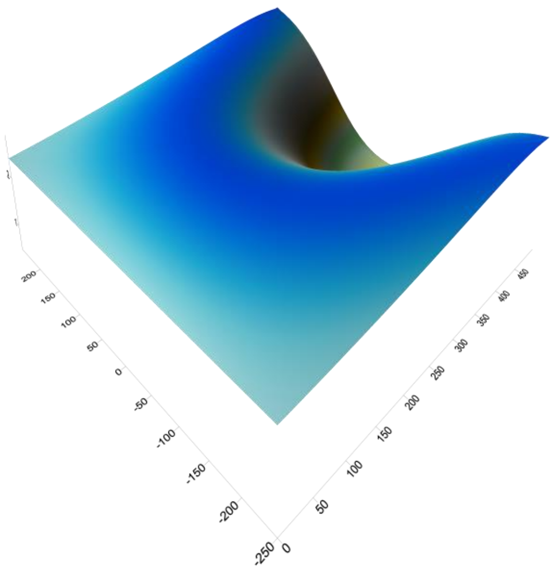

Fig.4: wave height 3-D Contour

In the analysis of refraction-diffraction results, a notable concentration of wave energy is observed at the center of the headland, coinciding with the point where breaking occurs. Both the shoaling-breaking model and the refraction-diffraction model incorporate equations involving the wave amplitude function, wave number conservation, and energy conservation to collectively simulate the processes of shoaling and breaking. These equations include weighting coefficients.

To determine suitable values for these weighting coefficients, an examination will be conducted using breaker index equations, as elaborated upon in the subsequent section..

IX. BREAKING INDEXES EQUATIONS.

In the preceding section, it has been demonstrated that the developed equations can effectively simulate shoaling and breaking. In this section, breaking indexes equations are formulated, and the role of corresponding weighting coefficients is studied.

9.1.Equation breaking.

The breaking equation is present in both the wave amplitude function Equation (15) and the wave constant Equation G (16). Breaking occurs when,

$$\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}-\frac{kA}{2}=0 \qquad \ldots\ldots(24)$$

Based on Equation (24), breaker indexes are formulated, including the breaker length index, breaker depth index, and breaker height index.

9.2. Breaker Length Index Equation

Equation (24) can be expressed as,

$$\frac{kA}{2}=\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}$$

Considering $k_x=\frac{k}{\sqrt{\gamma_x}}$ or $k=k_x\sqrt{\gamma_x}$ and $k_x=\frac{2\pi}{L_x}$ hence

$$\frac{H_b}{L_{b,x}}=\frac{2\tanh\theta\pi\sqrt{\gamma_x}}{\pi\sqrt{\gamma_z}} \qquad \ldots\ldots(25)$$

H_b represents the breaking wave height, while $L_{x,b}$ is the horizontal wavelength at the breaker point. In Equation (25), there are weighting coefficients, namely γ_x and γ_z, indicating the influence of weighting coefficients on breaking characteristics in this equation. In Table (3), the calculation of $\frac{H_b}{L_{b,x}}$ is presented for various values of ε and θ, where $\theta_1=1.60$, $\theta_2=1.70$, $\theta_3=1.80$ and $\theta_1=1.90$. The values of $\tanh\theta\pi$ include,

$\tanh 1.6\pi=0.999914$

$\tanh 1.7\pi=0.999954$

$\tanh 1.8\pi=0.999975$

$\tanh 1.9\pi=0.999987$

All of them ≈ 1.

Table.3 The value of $\frac{H_b}{L_{b,x}}$ on different ε and θ

ε	θ_1	θ_2	θ_3	θ_4
0.03	0.565	0.565	0.565	0.565
0.032	0.555	0.555	0.555	0.555
0.034	0.545	0.545	0.545	0.545
0.036	0.535	0.535	0.535	0.535
0.038	0.524	0.524	0.524	0.524
0.04	0.513	0.513	0.513	0.513
0.042	0.502	0.502	0.502	0.502
0.044	0.49	0.49	0.49	0.49
0.046	0.479	0.479	0.479	0.479

In Table (3), it can be observed that as the value of ε increases, the value of $\frac{H_b}{L_{b,x}}$ decreases. Meanwhile, concerning the deep water coefficient θ, the value of $\frac{H_b}{L_{b,x}}$ remains constant, as this is only considered up to the third decimal place. If examined up to the fifth decimal place, the $\frac{H_b}{L_{b,x}}$ value increases with an increase in the θ value.

9.3. Breaker Depth Index Equation.

Defined as

$$\tanh k\left(h+\frac{A}{2}\right)=\alpha k\left(h+\frac{A}{2}\right)$$

Substituting to (24),

$$\frac{\alpha k\left(h+\frac{A}{2}\right)}{\sqrt{\gamma_z}}-\frac{kA}{2}=0$$

$$\alpha k\left(h+\frac{A}{2}\right)-\frac{\sqrt{\gamma_z}kA}{2}=0$$

$$\alpha kh+\frac{\alpha kA}{2}-\frac{\sqrt{\gamma_z}kA}{2}=0$$

$$\alpha-\left(\sqrt{\gamma_z}-\alpha\right)\frac{A}{2h}=0$$

$k\left(h+\frac{A}{2}\right)=\theta\pi$ hence ,

Substituting to Equation α,

$$\alpha=\frac{\tanh\theta\pi}{\theta\pi}$$

At breaker point and on the sinusoidal wave, $A=\frac{H}{2}$ applies

$$\frac{\tanh\theta\pi}{\theta\pi}-\left(\sqrt{\gamma_z}-\frac{\tanh\theta\pi}{\theta\pi}\right)\frac{A}{2h}=0$$

$$\tanh\theta\pi-\left(\theta\pi\sqrt{\gamma_z}-\tanh\theta\pi\right)\frac{H_b}{4h_b}=0$$

$$\frac{H_b}{h_b}=\frac{4\tanh\theta\pi}{\left(\theta\pi\sqrt{\gamma_z}-\tanh\theta\pi\right)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(26)$$

This equation represents the breaker depth index, where h_b is the breaker depth. In equation (26), there is a weighting coefficient denoted as γ_z. Table (4) presents the results of the calculation of equation (26) for various values of θ and several values of ε. In the table, $\theta_1=1.60$, $\theta_2=1.70$, $\theta_3=1.80$ and $\theta_1=1.90$.

Table.4 the value of $\frac{H_b}{h_b}$ *on several values of* ε *and* θ

ε	θ_1	θ_2	θ_3	θ_4
0.03	0.869	0.808	0.754	0.708
0.032	0.853	0.793	0.741	0.695
0.034	0.836	0.777	0.726	0.682
0.036	0.819	0.762	0.712	0.668
0.038	0.801	0.745	0.697	0.654
0.04	0.783	0.728	0.681	0.639
0.042	0.764	0.711	0.665	0.625
0.044	0.745	0.694	0.649	0.61
0.046	0.726	0.676	0.633	0.594

The results in Table (4) indicate that the value of $\frac{H_b}{h_b}$ decreases with an increase in ε and also decreases with an increase in θ. For a constant value of H_b, the decrease in $\frac{H_b}{h_b}$ with an increase in θ suggests a deeper breaker depth. According to Mc. Cowan's criteria (1894), where $\frac{H_b}{h_b} = 0.78$, and considering the calculation of the breaker height h_b in subsection (9.5), resulting in $\varepsilon = 0.038 - 0.046$, the appropriate θ value is therefore 1.60.

9.4 Breaker depth-length index.

Equation (25) is written as an equation for H_b and substituted into (26), resulting in the breaker depth-length index equation, denoted as Equation (27).

$$\frac{h_b}{L_{b,x}} = \left(\frac{\theta}{2} - \frac{\tanh\theta\pi}{2\pi\sqrt{\gamma_z}}\right)\sqrt{\gamma_x} \quad \text{….(27)}$$

Condition $\frac{h}{L} \geq 1$ represents deep water conditions. Breaking due to bathymetry occurs in shallow water where $\frac{h}{L} < 1$ hence, (27) must be less than 1. Table (5) illustrates the values of $\frac{h_b}{L_{b,x}}$ for $= 1.60$ and $\theta = 2.30$

Table.5 Provides an overview of the values of $\frac{h_b}{L_b}$ *concerning the deep water coefficient* θ.

ε	$\theta = 1.60$		$\theta = 2.30$	
	$\frac{h_b}{L_{b,x}}$	$\frac{H_b}{h_b}$	$\frac{h_b}{L_{b,x}}$	$\frac{H_b}{h_b}$
0.03	0.65	0.869	0.996	0.567
0.032	0.651	0.853	0.997	0.557
0.034	0.652	0.836	0.997	0.547
0.036	0.653	0.819	0.998	0.536
0.038	0.655	0.801	0.998	0.525
0.04	0.656	0.783	0.999	0.514
0.042	0.657	0.764	0.999	0.502
0.044	0.658	0.745	1	0.491
0.046	0.659	0.726	1	0.479

From Table (5), it is evident that as the value of ε increases, the value of $\frac{h_b}{L_{b,x}}$, also increases, and as θ increases, the value of $\frac{h_b}{L_{b,x}}$ also increases. At $\theta = 2.30$, the value of $\frac{h_b}{L_{b,x}}$ approaches 1. It can be concluded that the value of θ must be less than 2.30.

9.5. Breaker Height Index Equation, $\frac{H_b}{H_0}$.

The breaker height index is the ratio of the breaker height to the wave height in deep water, denoted as $\frac{H_b}{H_0}$. The wave energy for a single wavelength is given by the equation:

$$E = c_E \rho g H^2 L$$

where c_E is the energy coefficient. In the linear wave theory (Dean (1991)), $c_E = \frac{1}{8}$. In this context, the value of c_E is not influential as it cancels out in the energy conservation equation. By equating the wave energy at the breaker point to the wave energy in deep water, we obtain Equation (28):

$$H_b^2 L_b = H_0^2 L_0 \quad \text{……(28)}$$

Equation (25) can be expressed as an equation for L_b and substituted into (28). Meanwhile, L_0 is substituted with (21), resulting in Equation (29):

$$\frac{H_b}{H_0} = \left(\frac{2\sqrt{\gamma_x}\left(\gamma_{t,2} + \frac{\gamma_z \gamma_{t,3}}{2}\right)}{\gamma_{t,3}\sqrt{\gamma_z}}\right)^{1/3} \quad \text{….(29)}$$

In this equation, there are no parameters for the deep water coefficient θ and wave period. Therefore, the breaker height is solely determined by the deep water wave height H_0 and the weighting coefficient.

To illustrate the influence of the weighting coefficient on the breaker height index $\frac{H_b}{H_0}$ and the breaker height H_b, the results are presented in Table (6). The calculated breaker height is compared with the breaker height from Komar, Paul D., and Gaughan, Michael K. (1968):

$$H_b = 0.39\, g^{1/5} (T H_0)^{2/5} \text{ m.}$$

In this equation, the wave period T, is set to 8 seconds, and two different deep water wave heights H_0 of 2.00 m and 2.40 m are used. In Table (6), H_{b-29} is the result of Equation (29) multiplied by H_0.

In Table (6), an observable trend is noted: as ε increases, the breaker height decreases. Following the Komar-Gaughan equation, a breaker height $H_0 = 2.00\, m$ is achieved at $\varepsilon =$

0.038, while for a higher initial wave height $H_0 = 2.40\ m$, the corresponding $\varepsilon = 0.046$. However, it is challenging to precisely determine ε from these findings. For a wave height of 2.40 m and a wave period of 8 seconds, identified as the maximum in that period, a pragmatic approach suggests using $\varepsilon = 0.042$ for wave heights below the maximum.

Table.6 Presents the calculated breaker height H_b.

ε	$\frac{H_b}{H_0}$	$H_0 = 2.0\ m$		$H_0 = 2.4\ m$	
		H_{b-29}	H_{b-K}	H_{b-29}	H_{b-K}
0.03	1.267	2.534	2.463	3.041	2.85
0.032	1.259	2.518	2.463	3.021	2.85
0.034	1.25	2.501	2.463	3.001	2.85
0.036	1.241	2.483	2.463	2.979	2.85
0.038	1.232	2.464	2.463	2.957	2.85
0.04	1.222	2.444	2.463	2.933	2.85
0.042	1.212	2.424	2.463	2.909	2.85
0.044	1.202	2.403	2.463	2.884	2.85
0.046	1.191	2.382	2.463	2.858	2.85

The study of breaker indexes underscores the substantial impact of weighting coefficients on breaking parameters. Consequently, employing the weighted Laplace equation in modeling wave dynamics is anticipated to yield more accurate and reliable results for breaking parameters.

The results of this study also indicate that the appropriate deep water coefficient is $\theta = 1.60$, with a value of $\varepsilon = 0.042$, along with weighting coefficients:

ε	$\gamma_{t,2}$	$\gamma_{t,3}$	γ_x	γ_z
0.042	1.99626	3.1097	0.97831	1.26448

X. CONCLUSION

The Weighted Taylor series refers to a truncated Taylor series that is transformed into a first-order series with significantly reduced truncation errors, and in some cases, without any errors. Utilizing the weighted Taylor series for formulating the continuity equation and Laplace equation results in the creation of the weighted continuity equation and weighted Laplace equation, characterized by minimal truncation errors. Consequently, constructing a water wave model using the weighted Laplace equation enhances the model's accuracy.

The pivotal role of weighting coefficients cannot be overstated, as they play a crucial role in determining wave characteristics such as wavelength and breaking parameters, including breaker height, depth, and length. This underscores the importance of utilizing the weighted Laplace equation in water wave modeling. Furthermore, since water wave modeling is inherently based on velocity potential, employing the weighted Laplace equation is deemed more effective.

In situations where the latest data from physical model research is available and reliable, the values of weighting coefficients can be easily adjusted to fine-tune the model. This adaptability enhances the applicability and accuracy of the water wave model, making it a valuable tool in studying water wave behavior.

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