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Fixed Point Results for Single-valued Mappings on a Set with Two Metrics using a Dass Gupta-type Bilateral Contraction

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Abstract*— The purpose of this paper is to explore some new fixed point results using a bilateral contraction. The first thing we need to do is recall the work on fixed-point results that have been done in different research papers. By combining the results of two papers, the first was by Rus [12], which discussed different fixed point results on a set with two metrics, and the second was by Chen [4], which used bilateral contractions to prove different fixed point results. In this paper, we present new results for single-valued mappings on a single set with two metrics. In order to accomplish all of this, a bilateral contraction of the type used by Dass Gupta has been used.

NTRODUCTION AND PRELIMINARIES

In a variety of branches of mathematics, fixed point theory provides important aspects to solve problems. During the last five decades, fixed point theory has grown in popularity [1]. A metric space is a non-empty set with metric (or distance function) defined on it. There is much use of metric spaces in different fields and applications, so it is expanded in many ways [2] [3] [5] [9] [16]. In [6] Zhang and Huang explained cone metric spaces. They briefly explained Banach's fixed point theorem for such spaces. Banach's fixed point theorem explains the conditions for the uniqueness of fixed points.

Maia [10], in 1968 investigated the famous result of the Banach contraction principle using two metrics on a non-empty set. Iseki [7], in 1975 described a fixed point theorem in a metric space. Rus [12], in 1977 proved a fixed point theorem in a set containing two metrics. Sigh and Pant [14], in 1981 proved a fixed point theorem in two metrics. Kaneko and Sessa [8], in 1989, established an idea about a fixed point theorem for contractive single and multivalued mappings. Takahashi [13], in 1996, introduced a fixed point of the multivalued mappings in convex metric spaces. Muresan [11], in 2007, gave some results about the fixed point theorem of Maia and expressed how to use these results in the sets with two metrics. Joonaghany and Karapinar [4], in 2019, enhanced the composition by combining the execution of results of two bilateral contractions; which includes Dass Gupta-type bilateral contraction. Stinson, Almuthaybiri and Tisdell [15], in 2020, described a notation about the development of fixed point theorems in a set containing two metrics with the help of iterated method.

As we begin, we define a Dass Gupta-type bilateral contraction, which is cited in a well-known paper by Chen [4].

Definition 1. Let $(S\,,\rho)$ be a non-empty set. The function $F : S \to S$ is called Dass Gupta-type bilateral contraction, if there is a $\phi : S \to [0\,,\infty)$ such that for all distinct $u, v \in S$

$$\rho(u\,,Fu) > 0$$

implies

$$\rho(Fu\,,Fv) \le [\phi(u) - \phi(Fu)] \cdot \max\left\{\rho(u\,,v)\,,\frac{[1+\rho(u\,,Fu)]\cdot\rho(v\,,Fv)}{1+\rho(u\,,v)}\right\}$$

Firstly, suppose that $\max\{\rho(u\,,v)\,,\rho(v\,,Fv)\} = \rho(u\,,v)$ then take a set with two metrics, and we make the new result, which is:

Theorem 1. Let S be a non-empty set. Suppose ρ_1 and ρ_2 be two metrics on S and $F : (S\,,\rho_1) \to (S\,,\rho_1)$ be a function. If there is a $\phi : S \to [0\,,\infty)$ and for all $u, v \in S$

(a) $\rho_1(Fu\,,Fv) \le [\phi(u) - \phi(Fu)] \cdot \rho_2(u\,,v)$
(b) $(S\,,\rho_1)$ is a complete metric space
(c) $F : (S\,,\rho_1) \to (S\,,\rho_1)$ is continuous
(d) $\exists\ \mu \in (0\,,1)$ we have $\rho_2(Fu\,,Fv) \le \mu \cdot \rho_2(u\,,v)$

Then F has a unique fixed point.

Proof. We prove the theorem by the iterative method. For any $u \in S$, let

$$u_0 = u$$
$$u_1 = Fu_0$$
$$u_2 = Fu_1$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$u_p = Fu_{p-1}$$

where $p \in \mathbb{N}$.

This implies that $\{u_p\}$ converges in S.

If $u_p = Fu_p$ then our theorem has been proved.

Suppose $u_p \neq Fu_p$. Then for any distinct $u_{p-1}\,, u_p \in S$, let $\tau_p = \rho_1(u_{p-1}\,,u_p)$ then by the given condition

$$\begin{aligned}\tau_{p+1} &= \rho_1(u_p\,,u_{p+1})\\ &= \rho_1(Fu_{p-1}\,,Fu_p)\\ &\le [\phi(u_{p-1}) - \phi(Fu_{p-1})] \cdot \rho_2(u_{p-1}\,,u_p)\\ &= [\phi(u_p - 1) - \phi(u_p)] \cdot \rho_2(u_{p-1}\,,u_p)\end{aligned}$$

It follows

$$\frac{\rho_1(u_1\,,u_{p+1})}{\rho_2(u_{p-1}\,,u_p)} \le \phi(u_{p-1}) - \phi(u_p)$$

$$0 < \frac{\rho_1(u_p\,,u_{p+1})}{\rho_2(u_{p-1}\,,u_p)} \le \phi(u_{p-1}) - \phi(u_p)$$

$$0 < \phi(u_{p-1}) - \phi(u_p)$$

$$\phi(u_{p-1}) > \phi(u_p)$$

We conclude that the sequence $\{\phi(u_p)\}$ is not only strictly decreasing but also necessarily positive. So $\{\phi(u_p)\}$ converges to some limit $l \geq 0$.

Now for each $p \in \mathbb{N}$ we have

$$\sum_{i=1}^{p} \frac{\rho_1(u_i\,, u_{i+1})}{\rho_2(u_{i-1}\,, u_i)} \leq \sum_{i=1}^{p} [\phi(u_{i-1}) - \phi(u_i)]$$
$$\leq [\phi(u_0) - \phi(u_1)] + [\phi(u_1) - \phi(u_2)] + \cdots + [\phi(u_{p-1}) - \phi(u_p)]$$
$$\leq \phi(u_0) - \phi(u_1) + \phi(u_1) - \phi(u_2) + \phi(u_2) + \cdots - \phi(u_{p-1}) + \phi(u_{p-1}) - \phi(u_p)$$
$$\leq \phi(u_0) - \phi(u_p)$$

If $p \to \infty$ then $\phi(u_p) \to l$

$$\sum_{i=1}^{p} \frac{\rho_1(u_i\,, u_{i+1})}{\rho_2(u_{i-1}\,, u_i)} \leq \phi(u_0) - l < \infty$$

In other words, we can say $\sum_{i=1}^{\infty} \frac{\rho_1(u_p\,, u_{p+1})}{\rho_2(u_{p-1}\,, u_p)}$ is a finite positive number.

By induction, $\frac{\rho_1(u_p\,, u_{p+1})}{\rho_2(u_{p-1}\,, u_p)}$ is bounded in $(0\,, 1)$, then there exists some $\mu \in (0\,, 1)$ we have

$$\frac{\rho_1(u_p\,, u_{p+1})}{\rho_2(u_{p-1}\,, u_p)} \leq \mu$$

$$\rho_1(u_p\,, u_{p+1}) \leq \mu \cdot \rho_2(u_{p-1}\,, u_p)$$
$$\leq \mu^2 \cdot \rho_2(u_{p-2}\,, u_{p-1})$$
$$\leq \mu^3 \cdot \rho_2(u_{p-3}\,, u_{p-2})$$
$$\dots \dots \dots$$
$$\dots \dots \dots$$
$$\dots \dots \dots$$
$$\leq \mu^p \cdot \rho_2(u_0\,, u_1)$$

Now, for each $p, q \in \mathbb{N}$ with $p < q$ such that

$$\rho_1(u_p\,, u_q) \leq [\phi(u_p) - \phi(Fu_p)] \cdot \rho_2(u_{p-1}\,, u_{q-1})$$
$$\leq [\phi(u_p) - \phi(u_{p+1})]\mu \cdot \rho_2(u_{p-2}\,, u_{q-2})$$
$$\leq [\phi(u_p) - \phi(u_{p+1})]\mu^2 \cdot \rho_2(u_{p-3}\,, u_{q-3})$$
$$\dots \dots \dots$$
$$\dots \dots \dots$$
$$\dots \dots \dots$$
$$\leq [\phi(u_p) - \phi(u_{p+1})]\mu^{p-1} \cdot \rho_2(u_0\,, u_{q-p})$$

Since, $\phi(u_p)$ is strictly decreasing, then $[\phi(u_p) - \phi(u_{p+1})]$ is very small and $\mu \in (0\,, 1)$ then we can conclude that $[\phi(u_p) - \phi(u_{p+1})]\mu^{p-1} < \epsilon$ then

$$\rho_1(u_p\,, u_q) < \epsilon \cdot \rho_2(u_0\,, u_{q-p})$$
$$< \epsilon$$

This implies that $\{u_p\}$ is the Cauchy sequence.

Since S is complete. By the continuity of $: (S\,, \rho_1) \to (S\,, \rho_1)$, for any $u_0 \in S$

$$
\begin{aligned}
u_0 &= \lim_{p\to\infty}[F^p(u_0)] \\
&= \lim_{p\to\infty}[F.F^{p-1}(u_0)] \\
&= F\left(\lim_{p\to\infty}[F^{p-1}(u_0)]\right) \\
&= F(u_0)
\end{aligned}
$$

Thus, $u_0 \in S$ is a fixed point of F.

Suppose $v_0 \in S$ is another fixed point of F, then

$$
\begin{aligned}
\rho_2(u_0\,,v_0) &= \rho(Fu_0\,,Fv_0) \\
&\le u \cdot \rho_2(u_0\,,v_0) \\
(1-\mu)\cdot\rho_2(u_0\,,v_0) &\le 0 \\
\rho_2(u_0\,,v_0) &= 0 \\
u_0 &= v_0
\end{aligned}
$$

Hence, u_0 is a unique fixed point of F.

By applying some more conditions to the above theorem, we make a new result. More conditions were taken from the paper by Rus [12].

Theorem 2. Let S be a non-empty set. Suppose ρ_1 and ρ_2 be two metrics on S and $F\,,F_p : S \to S$ be the functions. If for all $u\,,v \in S$ such that

(a) $(S\,,\rho_1)\,,(S\,,\rho_2)$ and F satisfy the hypothesis of Theorem 1

(b) The sequence F_p uniformly converges on $(S\,,\rho_1)$ to F

(c) $\exists\ \lambda > 0$ we have $\rho_2(u\,,v) \le \lambda \cdot \rho_1(u\,,v)$

Then for every $u_p \in S$, sequence $\{u_p\}$ converges to a unique fixed point u_0 of F.

Proof. We prove that every sequence $\{u_p\} \subseteq S$ converges to a unique fixed point $u_0 \in S$. Since for some $p \in \mathbb{N}$

$$F^p(u_p) = u_p$$

Now,

$$
\begin{aligned}
\rho_1(u_p\,,u_0) &= \rho_1\left(F_p^2(u_p)\,,F^2(u_0)\right) \\
&\le \rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right) + \rho_1\left(F^2(u_p)\,,F^2(u_0)\right) \\
&\le \rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right) + [\phi(u_p)-\phi(Fu_p)]\cdot\rho_2\left(F^2(u_p)\,,F^2(u_0)\right) \\
&\le \rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right) + [\phi(u_p)-\phi(Fu_p)]\mu\cdot\rho_2(u_p\,,u_0) \\
&\le \rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right) + [\phi(u_p)-\phi(Fu_p)]\mu\lambda\cdot\rho_1(u_p\,,u_0)
\end{aligned}
$$

Since

$$
\begin{aligned}
&[\phi(u_p)-\phi(Fu_p)]\mu\lambda < 1 \\
&=> [\phi(u_p)-\phi(Fu_p)]\mu\lambda \to 0 \\
&=> [\phi(u_p)-\phi(Fu_p)]\mu\lambda < \epsilon
\end{aligned}
$$

Then

$$\rho_1(u_p\,,u_0) \le \rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right) + \epsilon\cdot\rho_1(u_p\,,u_0)$$

$$(1-\epsilon)\cdot\rho_1(u_p\,,u_0)\leq\rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right)$$

$$\rho_1(u_p\,,u_0)\leq(1-\epsilon)^{-1}\cdot\rho_1\left(F_p^2(u_p)\,,F^2(u_p)\right)$$

$$\leq(1-\epsilon)^{-1}\cdot\left[\rho_1\left(F_p^2(u_p)\,,F.F_p(u_p)\right)+\rho_1\left(F.F_p(u_p)\,,F^2(u_p)\right)\right]$$

$$\leq(1-\epsilon)^{-1}\cdot\left[\rho_1\left(F_p^2(u_p)\,,F.F_p(u_p)\right)+\epsilon_1\cdot\rho_1\left(F_p(u_p)\,,F(u_p)\right)\right]$$

It is given that F_p uniformly converges to F in metric ρ_1 , then $\rho_1\left(F_p^2(u_p)\,,F\cdot F_p(u_p)\right)\to 0$ and $\rho_1\left(F_p(u_p)\,,F(u_p)\right)\to 0$ as $p\to\infty$. Thus

$$\rho_1(u_p\,,u_0)\leq(1-\epsilon)^{-1}\cdot\left[\rho_1\left(F_p^2(u_p)\,,F\cdot F_p(u_p)\right)+\epsilon_1\cdot\rho_1\left(F_p(u_p)\,,F(u_p)\right)\right]\to 0$$

It means

$$\rho_1(u_p\,,u_0)\to 0$$

as $p\to\infty$.

Hence, $\{u_p\}$ converges in $(S\,,\rho_1)$ to a unique fixed point u_0 of F.

Now, suppose in the definition-1, if $\max\{\rho(u\,,v)\,,\rho(v\,,Fv)\}=\frac{[1+\rho(u\,,Fu)]\cdot\rho(v\,,Fv)}{1+\rho(u\,,v)}$ then one more new result is generated.

Theorem 3 Let S be a non-empty set. Suppose ρ_1 and ρ_2 be two metrics on S and $F:(S\,,\rho_1)\to(S\,,\rho_1)$ be a function. If there is a $\phi:S\to[0\,,\infty)$ and for all $u,v\in S$ such that

(a) $\rho_1(Fu\,,Fv)\leq[\phi(u)-\phi(Fu)]\cdot\frac{[1+\rho_2(u\,,Fu)]\cdot\rho_2(v\,,Fv)}{1+\rho_2(u\,,v)}$
(b) $(S\,,\rho_1)$ is a complete metric space
(c) $F:(S\,,\rho_1)\to(S\,,\rho_1)$ is continuous
(d) $\exists\ \mu\in(0\,,1)$ we have $\rho_2(Fu\,,Fv)\leq\mu\cdot\rho_2(u\,,v)$

Then F has a unique fixed point.

Proof. We prove the theorem by the iterative method. For any $u\in S$, let

$$u_0=u$$
$$u_1=Fu_0$$
$$u_2=Fu_1$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$\dots\dots\dots$$
$$u_p=Fu_{p-1}$$

where $p\in\mathbb{N}$.

This implies that $\{u_p\}$ converges in S.

If $u_p=Fu_p$ then our theorem has been proved.

Suppose $u_p\neq Fu_p$. Then for any distinct $u_{p-1}\,,u_p\in S$, let $\tau_p=\rho_1(u_{p-1}\,,u_p)$ then by the given condition

$$\tau_{p+1}=\rho_1(u_p\,,u_{p+1})$$

$$=\rho_1(Fu_{p-1}\,,Fu_p)$$

$$\leq[\phi(u_{p-1}-\phi(Fu_{p-1})]\cdot\frac{[1+\rho_2(u_{p-1}\,,Fu_{p-1})]\cdot\rho_2(u_p\,,Fu_p)}{1+\rho_2(u_{p-1}\,,u_p)}$$

$$\leq [\phi(u_{p-1}) - \phi(u_p)] \cdot \frac{[1+\rho_2(u_{p-1}, u_p)] \cdot \rho_2(u_p, u_{p+1})}{1+\rho_2(u_{p-1}, u_p)}$$

$$\leq [\phi(u_{p-1}) - \phi(u_p)] \cdot \rho_2(u_p, u_{p+1})$$

It follows

$$\frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_p, u_{p+1})} \leq \phi(u_{p-1}) - \phi(u_p)$$

$$0 < \frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_p, u_{p+1})} \leq \phi(u_{p-1}) - \phi(u_p)$$

$$0 < \phi(u_{p-1}) - \phi(u_p)$$

$$\phi(u_{p-1}) > \phi(u_p)$$

We conclude that the sequence $\{\phi(u_p)\}$ is not only strictly decreasing but also necessarily positive. So $\{\phi(u_p)\}$ converges to some limit $l \geq 0$.

Now for each $p \in \mathbb{N}$ we have

$$\sum_{i=1}^{p} \frac{\rho_1(u_i, u_{i+1})}{\rho_2(u_i, u_{i+1})} \leq \sum_{i=1}^{p} [\phi(u_{i-1}) - \phi(u_i)]$$

$$\leq [\phi(u_0) - \phi(u_1)] + [\phi(u_1) - \phi(u_2)] + \cdots + [\phi(u_{p-1}) - \phi(u_p)]$$

$$\leq \phi(u_0) - \phi(u_1) + \phi(u_1) - \phi(u_2) + \phi(u_2) + \cdots - \phi(u_{p-1}) + \phi(u_{p-1}) - \phi(u_p)$$

$$\leq \phi(u_0) - \phi(u_p)$$

If $p \to \infty$ then $\phi(u_p) \to l$

$$\sum_{i=1}^{p} \frac{\rho_1(u_i, u_{i+1})}{\rho_2(u_i, u_{i+1})} \leq \phi(u_0) - l < \infty$$

In other words, we can say $\sum_{i=1}^{\infty} \frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_p, u_{p+1})}$ is a finite positive number.

By induction, $\frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_p, u_{p+1})}$ is bounded in $(0, 1)$, then there exists some $\mu \in (0, 1)$ we have

$$\frac{\rho_1(u_p, u_{p+1})}{\rho_2(u_p, u_{p+1})} \leq \mu$$

$$\rho_1(u_p, u_{p+1}) \leq \mu \cdot \rho_2(u_p, u_{p+1})$$

$$\leq \mu^2 \cdot \rho_2(u_{p-1}, u_p)$$

$$\leq \mu^3 \cdot \rho_2(u_{p-2}, u_{p-1})$$

$$\dots \dots \dots$$

$$\dots \dots \dots$$

$$\dots \dots \dots$$

$$\leq \mu^{p+1} \cdot \rho_2(u_0, u_1)$$

Now, for each $p, q \in \mathbb{N}$ with $p < q$ such that

$$\rho_1(u_p, u_q) \leq [\phi(u_p) - \phi(Fu_p)] \cdot \frac{[1+\rho_2(u_p, Fu_q)] \cdot \rho_2(u_p, Fu_q)}{1+\rho_2(u_p, u_q)}$$

$$\leq [\phi(u_p) - \phi(u_{p+1})] \cdot \frac{[1+\rho_2(u_p, u_{q+1})] \cdot \rho_2(u_p, u_{q+1})}{1+\rho_2(u_p, u_q)}$$

$$\leq [\phi(u_p) - \phi(u_{p+1})]\mu \cdot \rho_2(u_p , u_{q+1})$$
$$\leq [\phi(u_p) - \phi(u_{p+1})]\mu^2 \cdot \rho_2(u_{p-1} , u_q)$$
$$\dots \dots \dots$$
$$\dots \dots \dots$$
$$\dots \dots \dots$$
$$\leq [\phi(u_p) - \phi(u_{p+1})]\mu^p \cdot \rho_2(u_1 , u_{q-p+2})$$
$$\leq [\phi(u_p) - \phi(u_{p+1})]\mu^{p+1} \cdot \rho_2(u_0 , u_{q-p+1})$$

Since, $\phi(u_p)$ is strictly decreasing, then $[\phi(u_p) - \phi(u_{p+1})]$ is very small and $\mu \in (0 , 1)$ then we can conclude that $[\phi(u_p) - \phi(u_{p+1})]\mu^{p+1} < \epsilon$ then

$$\rho_1(u_p , u_q) < \epsilon \cdot \rho_2(u_0 , u_{q-p+1})$$
$$< \epsilon$$

This implies that $\{u_p\}$ is the Cauchy sequence.

Since S is complete. By the continuity of $: (S , \rho_1) \to (S , \rho_1)$, for any $u_0 \in S$

$$u_0 = \lim_{p\to\infty} [F^p(u_0)]$$
$$= \lim_{p\to\infty} [F.F^{p-1}(u_0)]$$
$$= F\left(\lim_{p\to\infty} [F^{p-1}(u_0)]\right)$$
$$= F(u_0)$$

Thus, $u_0 \in S$ is a fixed point of F.

Suppose $v_0 \in S$ is another fixed point of F, then

$$\rho_2(u_0 , v_0) = \rho(Fu_0 , Fv_0)$$
$$\leq u \cdot \rho_2(u_0 , v_0)$$
$$(1 - \mu) \cdot \rho_2(u_0 , v_0) \leq 0$$
$$\rho_2(u_0 , v_0) = 0$$
$$u_0 = v_0$$

Hence, u_0 is a unique fixed point of F.

Similarly, by applying some more conditions to the above result, we make a new result.

Theorem 4. Let S be a non-empty set. Suppose ρ_1 and ρ_2 be two metrics on S and $F , F_p : S \to S$ be the functions. If for all $u , v \in S$ such that

(a) $(S , \rho_1) , (S , \rho_2)$ and F satisfy the hypothesis of Theorem 3
(b) The sequence F_p uniformly converges on (S , ρ_1) to F
(c) $\exists \ \lambda > 0$ we have $\rho_2(Fu , Fv) \leq \lambda \cdot \rho_1(u , v)$

Then for every $u_p \in S$, sequence $\{u_p\}$ converges to a unique fixed point u_0 of F.

Proof. We prove that every sequence $\{u_p\} \subseteq S$ converges to a unique fixed point $u_0 \in S$. Since for some $p \in \mathbb{N}$

$$F^p(u_p) = u_p$$

Now,

$$\rho_1(u_p , u_0) = \rho_1\left(F_p^2(u_p) , F^2(u_0)\right)$$

$$\leq \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + \rho_1\left(F^2(u_p), F^2(u_0)\right)$$
$$\leq \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + [\phi(u_p) - \phi(u_{p+1})] \cdot \rho_2\left(F^2(u_p), F^2(u_0)\right)$$
$$\leq \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + [\phi(u_p) - \phi(u_{p+1})]\mu \cdot \rho_2(Fu_p, Fu_0)$$
$$\leq \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + [\phi(u_p) - \phi(u_{p+1})]\mu\lambda \cdot \rho_1(u_p, u_0)$$

Since $\phi(u_p)$ is strictly decreasing then $[\phi(u_p) - \phi(u_{p+1})]$ is very small and $\mu \in (0, 1)$ then we conclude that $[\phi(u_p) - \phi(u_{p+1})]\mu\lambda < \epsilon$ then

$$\rho_1(u_p, u_0) \leq \rho_1\left(F_p^2(u_p), F^2(u_p)\right) + \epsilon \cdot \rho_1(u_p, u_0)$$
$$(1-\epsilon) \cdot \rho_1(u_p, u_0) \leq \rho_1\left(F_p^2(u_p), F^2(u_p)\right)$$
$$\rho_1(u_p, u_0) \leq (1-\epsilon)^{-1} \cdot \rho_1\left(F_p^2(u_p), F^2(u_p)\right)$$
$$\leq (1-\epsilon)^{-1} \cdot \left[\rho_1\left(F_p^2(u_p), F.F_p(u_p)\right) + \rho_1\left(F.F_p(u_p), F^2(u_p)\right)\right]$$
$$\leq (1-\epsilon)^{-1} \cdot \left[\rho_1\left(F_p^2(u_p), F.F_p(u_p)\right) + \epsilon_1 \cdot \rho_1\left(F_p(u_p), F(u_p)\right)\right]$$

It is given that F_p uniformly converges to F in metric ρ_1, then $\rho_1\left(F_p^2(u_p), F \cdot F_p(u_p)\right) \to 0$ and $\rho_1\left(F_p(u_p), F(u_p)\right) \to 0$ as $p \to \infty$. This implies that

$$\rho_1(u_p, u_0) \leq (1-\epsilon)^{-1} \cdot \left[\rho_1\left(F_p^2(u_p), F \cdot F_p(u_p)\right) + \epsilon_1 \cdot \rho_1\left(F_p(u_p), F(u_p)\right)\right] \to 0$$

It means

$$\rho_1(u_p, u_0) \to 0$$

as $p \to \infty$.

Hence, $\{u_p\}$ converges in (S, ρ_1) to a unique fixed point u_0 of F.

By the above theorems, we proved the new fixed point results on a set with two metrics using the idea of a bilateral contraction. Now, we will take an example, which helps us to prove the inequalities, which we used in the above results and disprove the other contraction inequalities.

Example 1. Let $S = \{0, 1, 2\}$ endowed with the metric ρ_1 and ρ_2 defined for all $u, v \in S$

$$\rho_1(u, v) = \begin{cases} 0 & if \quad u = v \\ 1 & if \quad u \neq v \end{cases} \qquad \text{and} \qquad \rho_2(u, v) = |u - v|$$

Let $F : S \to S$ defined by

$$F(0) = 0, F(1) = 2 \; and \; F(2) = 0$$

Define $\phi : S \to [0, \infty)$ as

$$\phi(0) = 0, \phi(1) = 4 \; and \; \phi(2) = 2$$

Prove that for all $, v \in S$, F satisfies

$$\rho_1(Fu, Fv) \leq [\phi(u) - \phi(Fu)] \cdot \rho_2(u, v)$$

We prove for all $u, v \in S$

(i) For $(u, v) = (0, 0)$:

$$\rho_1(F0, F0) \leq [\phi(0) - \phi(F0)] \cdot \rho_2(0, 0)$$
$$\rho_1(0, 0) \leq [\phi(0) - \phi1(0)] \cdot \rho_2(0, 0)$$

$$0 \le [4-2] \cdot |1-0|$$
$$1 \le 2$$

(iii) For $(u\,,v) = (1\,,1)$:

$$\rho_1(F1\,,F1) \le [\phi(1) - \phi(F1)] \cdot \rho_2(1\,,1)$$
$$\rho_1(2\,,2) \le [\phi(1) - \phi(2)] \cdot \rho_2(1\,,1)$$
$$0 \le [4-2] \cdot |1-1|$$
$$0 \le 0$$

(iv) For $(u\,,v) = (1\,,2)$:

$$\rho_1(F1\,,F2) \le [\phi(1) - \phi(F1)] \cdot \rho_2(1\,,2)$$
$$\rho_1(2\,,0) \le [\phi(1) - \phi(2)] \cdot \rho_2(1\,,2)$$
$$1 \le [4-2] \cdot |1-2|$$
$$1 \le 2$$

(v) For $(u\,,v) = (2\,,0)$:

$$\rho_1(F2\,,F0) \le [\phi(2) - \phi(F2)] \cdot \rho_2(2\,,0)$$
$$\rho_1(0\,,0) \le [\phi(2) - \phi(0)] \cdot \rho_2(2\,,0)$$
$$0 \le [2-0] \cdot |2-0|$$
$$0 \le 4$$

(vi) For $(u\,,v) = (2\,,2)$:

$$\rho_1(F2\,,F2) \le [\phi(2) - \phi(F2)] \cdot \rho_2(2\,,2)$$
$$\rho_1(0\,,0) \le [\phi(2) - \phi(0)] \cdot \rho_2(2\,,2)$$
$$0 \le [2-0] \cdot |2-2|$$
$$0 \le 0$$

Hence, for all $, v \in S$, F satisfied the given inequality.

Now, we check F doesn't satisfy other contraction inequalities. Suppose the contraction inequality on two metrics:

$$\rho_1(Fu\,,Fv) \le \lambda \cdot \rho_2(u\,,v)$$

for some $\lambda > 0$.

For $(u\,,v) = (1\,,2)$:

$$\rho_1(F1\,,F2) \le \lambda \cdot \rho_2(1\,,2)$$
$$1 \le \lambda \cdot |1-2|$$
$$1 \le \lambda \cdot |-1|$$
$$1 \le \lambda$$

This is false for $0 < \lambda < 1$. So

$$1 \nleq \lambda$$

Hence, F doesn't satisfy $\rho_1(Fu\,,Fv) \le \lambda \cdot \rho_2(u\,,v)$.

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