International Journal of Advanced Engineering Research and Science (IJAERS)
Peer-Reviewed Journal
ISSN: 2349-6495(P) | 2456-1908(O)
Vol-10, Issue-12; Dec, 2023
Journal Home Page Available: https://ijaers.com/
Article DOI: https://dx.doi.org/10.22161/ijaers.1012.11

Method for Determining Weighting Coefficients in Weighted Taylor Series Applied to Water Wave Modeling

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Received: 05 Nov 2023,
Receive in revised form: 12 Dec 2023,
Accepted: 20 Dec 2023,
Available online: 29 Dec 2023

***Keywords*— weighted Taylor series, weighting coefficients calculation, Weighted Laplace equation.**

***Abstract*— The Weighted Taylor series is an adaptation of the conventional Taylor series truncated to the first order, wherein high-order differential terms are replaced by introducing weighting coefficients to the initial terms. This study presents a novel approach for computing these weighting coefficients specifically designed for water wave modeling. Subsequently, the derived weighted Taylor series is employed to formulate both weighted continuity and the weighted Laplace equation. The weighted Laplace equation facilitates the formulation of the velocity potential equation, leading to the development of wave transformation equations encompassing important phenomena such as shoaling, breaking, and refraction-diffraction. Additionally, the formulation of the weighted Euler momentum conservation equation is introduced to determine the wave number in deep water. By scrutinizing the outcomes of dispersion equations, as well as analyzing shoaling-breaking and refraction-diffraction scenarios, optimal values for the weighting coefficients are identified..*

I. INTRODUCTION

Hydrodynamic equations are commonly expressed through a truncated Taylor series, where terms of order 2 and higher are omitted, reducing the series to include only order 1 differentials. This simplification is based on the assumption that, within small intervals such as, δt, δx, δz, the values of second-order terms and beyond become negligible and can be disregarded.

There is a lack of prior research systematically investigating the Taylor series truncation method. Hutahaean (2021) introduced the weighted Taylor series by examining intervals where the values of 1st order terms significantly surpass those of 2nd order terms. Building upon this, Hutahaean (2022) applied the Forward Difference scheme to formulate the weighted Taylor series, assigning a weighting coefficient solely to the time derivative term in the function $f(x,z,t)$. Both studies demonstrated that the weighting coefficient in the truncated Taylor series can alter wave characteristics, shortening wavelength and reducing water particle velocity. These findings suggest that utilizing a weighted Taylor series with optimized coefficients can enhance water wave models.

In this study, the Taylor series was truncated using the central-difference method. This method only extracts contributions from even-order differential terms, and is subsequently corrected by the remaining contributions from odd-order differential terms.

The truncated Taylor series was then utilized to formulate the foundational equation of hydrodynamics, specifically the continuity equation. Each term in this equation received its respective weighting coefficient, leading to the designation of a weighted continuity equation.

Employing the weighted continuity equation, the formulation of the weighted Laplace Equation followed. Subsequently, the weighted Laplace equation was solved for sloping bottoms to derive conservation equations governing variations in wave constants as waves transition from deeper to shallower waters.

By utilizing these conservation equations, the study derived equations describing changes in wave constants, including wave amplitude and wavelength in shallower waters. These equations incorporate weighting coefficients, with the corresponding values of the weighting coefficients studied based on the results of dispersion equation, shoaling breaking model, and refraction-diffraction model.

II. WEIGHTED TAYLOR SERIES

Taylor series for a function with two variables, $f = f(x,t)$, (Arden, Bruce W. and Astill Kenneth N. ,1970) is

$$f(x+\delta x, t+\delta t) = f(x,t) + +\delta t\frac{\partial f}{\partial t} + \delta x\frac{\partial f}{\partial x} \quad \frac{\delta t^2}{2!}\frac{\partial^2 f}{\partial t^2} + \delta t\delta x\frac{\partial^2 f}{\partial t\partial x} + \frac{\delta x^2}{2!}\frac{\partial^2 f}{\partial x^2} + \cdots \qquad (1)$$

In this paper, x represents the horizontal axis, z the vertical axis, and t time. In the formulation of hydrodynamic equations, including continuity equation and Euler's momentum conservation equation, this Taylor series is truncated to a first-order differential series, assuming that within the interval δt, δx are very small, higher-order differential terms are very small and can be neglected, i.e., it becomes:.

$$f(x+\delta x, z+\delta z, t+\delta t) = f(x,z,t) + \delta t\frac{\partial f}{\partial t} + \delta x\frac{\partial f}{\partial x}$$

However, as the interval decreases, not only do the values of higher-order differential terms decrease, but the values of first-order terms also decrease. As a result, the relative values of higher-order terms to the first-order terms remain significant.

Equation (1) can be written as follows,

$$f(x+\delta x, t+\delta t) = f(x,t) + \left(1 + \frac{\delta t}{2}\frac{\partial}{\partial t} + \delta x\frac{\partial}{\partial x} + ..\right)\delta t\frac{\partial f}{\partial t} + \left(1 + \frac{\delta x}{2}\frac{\partial}{\partial x} + ..\right)\delta x\frac{\partial f}{\partial x}$$

There are contributions from higher-order terms to the first-order terms. Therefore, these higher-order terms cannot be simply neglected. Truncating the series to the first order should be accompanied by providing coefficients that represent the higher-order terms.

$$f(x+\delta x, z+\delta z, t+\delta t) = f(x,z,t) + \gamma_t\delta t\frac{\partial f}{\partial t} + \gamma_x\delta x\frac{\partial f}{\partial x}$$

This series is a weighted Taylor series, with weighting coefficients γ_t and γ_x. With this equation, the role of higher-order terms is not completely eliminated; it is represented by the weighting coefficients. The characteristics of the function present in the higher-order terms are still reflected in the weighting coefficients.

The aim of this research is to develop a method for calculating weighting coefficients by extracting contributions from higher-order terms and adding them to the first-order terms.

The method of absorbing contributions from high-order terms or the formation of weighting coefficients generally consists of two parts:

a. Absorbing contributions from odd-order differential terms.

b. Absorbing contributions from even-order differential terms.

In this research, weighting coefficients will be formulated for the water wave modeling equation. The solution to the velocity potential equation of the Laplace equation (Dean (1991)) is:

$$\phi(x,z,t) = G\cos kx\cosh k(h+z)\sin\sigma t$$

k is wave number, $k = \frac{2\pi}{L}$, σ is the angular frequency, $\sigma = \frac{2\pi}{T}$, T is wave period and h is water depth. Considering that sin function has similar characteristics to the cos function, a method for calculating weighting coefficients will be developed using a functional form.,

$$f(x,z,t) = \cos kx\cosh k(h+z)\cos\sigma t$$

III. ABSORPTION OF CONTRIBUTIONS FROM THIRD-ORDER DIFFERENTIAL TERMS

In this research, a calculation method is developed wherein the Taylor series is employed up to the third order only. Limitations are introduced by utilizing an interval size where terms of the fourth order and higher become negligible. Subsequently, the third-order terms are removed from the series, and their values are added to the first-order terms. The magnitude of the contribution of third-order terms to the first-order terms is expressed with a contribution coefficient.

The contribution coefficient for the time differential terms is given by,

$$\mu_t = \frac{\frac{\delta t^3}{6}\frac{\partial^3 f}{\partial t^3}}{\delta t\frac{\partial f}{\partial t}}$$

As a function of time t is $f(t) = \cos\sigma t$, the differential substitution of this function, performed under the condition $\cos\sigma t = \sin\sigma t$

$$\mu_t = -\frac{\delta t^2}{6}\sigma^2$$

Substituting $\delta t = \varepsilon_t T$ and $\sigma = \frac{2\pi}{T}$,

$$\mu_t = -\frac{4\pi^2}{6}\varepsilon_t^2$$

ε_t is the time interval coefficient t. Similarly, for the function $f(x) = \cos kx$, the following is obtained

$$\mu_x = -\frac{4\pi^2}{6}\varepsilon_x^2$$

ε_x is the coefficient of the interval x, where $\delta x = \varepsilon_x L$, and

$$\mu_z = \frac{4\pi^2}{6}\varepsilon_z^2$$

ε_z is the coefficient of the interval z, where $\delta z = \varepsilon_z L$.

It is important to note that the computation of contribution coefficients necessitates values for the interval coefficients ε_t, ε_x and ε_z which will be discussed in the subsequent section.

3.1. Calculation of Interval Coefficients

In this section, the method of calculating the values of interval coefficients ε_t, ε_x and ε_z, is discussed. The calculation is performed using the optimization equation.

$$\left|\frac{s_2+s_3}{s_1}\right| \leq \varepsilon \qquad \text{………..(2)}$$

s_1, s_2 dan s_3, are, in order, the first, second, and third terms in the Taylor series. Term 1 is a term with a first-order differential, term 2 is a term with a second-order differential, and term 3 is a term with a third-order differential.

ε is a small number whose value is determined. The larger the value of ε, the larger the values of higher-order terms that will be extracted. Furthermore, ε is referred to as the optimization coefficient..

The optimization equation is an equation with variables representing the size of the interval, i.e., $\delta t, \delta x, \delta z$. This equation for the interval size is converted into an equation for the interval coefficients, i.e., ε_t, ε_x and ε_z, by substituting $\delta t = \varepsilon_t T$, $\delta x = \varepsilon_x L$ dan $\delta z = \varepsilon_z L$. T is the wave period, and L is the wavelength.

a. Calculation of Time Interval Coefficient at -t ε_t in the function $f(t) = \cos\sigma t$.

By utilizing a series up to order 3, the optimization equation can be expressed as,

$$\left|\frac{\frac{\delta t^2}{2}\frac{d^2 f}{dt^2}+\frac{\delta t^3}{6}\frac{d^3 f}{dt^3}}{\delta t\frac{df}{dt}}\right| = \varepsilon$$

Substituting the differentials of $f(t)$ when $\cos\sigma t = \sin\sigma t$, a second-degree polynomial for δt is obtained.

$$\frac{\delta t}{2}\sigma - \frac{\delta t^2}{6}\sigma^2 = \varepsilon$$

Substituting $\delta t = \varepsilon_t T$, and $\sigma = \frac{2\pi}{T}$, where T is wave period,

$$\frac{2\pi^2}{3}\varepsilon_t^2 - \pi\varepsilon_t + \varepsilon = 0$$

Of the two ε_t, the least value is used.

In the equation for ε_t there are neither variable σ or T. Hence, ε_t is not dependent to the wave period T and it applies to any wave period T.

b. Interval Coefficient -x ε_x on function $f(x,t) = \cos\sigma t \cos kx$.

For this function, there will be two variables in the optimization equation ε_t and ε_x. By using ε_t from the calculation results for the function $f(t) = \cos\sigma t$, hence ε_x is the only unknown variable.

The optimization equation for the function $f(x,t)$ is worked out using the Taylor series up to the third term, resulting in a third-degree polynomial. This polynomial can be solved using the Newton-Rhapson iteration method, which requires an initial iteration value. To obtain the initial iteration value, the optimization equation is first solved using only the second-order differential terms, forming a quadratic polynomial..

$$\left|\frac{s_2}{s_1}\right| \geq \varepsilon.$$

Where

$$s_1 = \delta t\frac{\partial f}{\partial t} + \delta x\frac{\partial f}{\partial x}$$

$$s_2 = \frac{\delta t^2}{2}\frac{\partial^2 f}{\partial t^2} + \delta t\delta x\frac{\partial^2 f}{\partial t\partial x} + \frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2}$$

Substituting s_1 and s_2 into the optimization equation, the computation proceeds under the conditions $\cos\sigma t = \sin\sigma t$ and $\cos kx = \sin kx$, resulting in the cancellation of terms in the numerator and denominator. Subsequently, the substitution $\delta t = \varepsilon_t T, \delta x = \varepsilon_x L, \sigma = \frac{2\pi}{T}$ and $k = \frac{2\pi}{L}$ with ε_t as a known variable, a quadratic equation in ε_t is derived from calculations involving the function $f(t) = \cos\sigma t$

$$c_0 + c_1\varepsilon_x + c_2\varepsilon_x^2 = 0$$

$$c_0 = 2\pi\varepsilon_t\varepsilon - 2\pi^2\varepsilon_t^2$$

$$c_1 = 2\pi\varepsilon + 4\pi^2\varepsilon_t$$

$$c_2 = -2\pi^2$$

The quadratic equation yields two roots, of which the larger one is selected. Notably, the ε_x \ equation no longer contains the wave period T or wavelength L.

Proceeding to the optimization equation (2), we obtain a cubic polynomial.

$$s_3 = \frac{\delta t^3}{6}\frac{\partial^3 f}{\partial t^3} + \frac{\delta t^2}{2}\delta x\frac{\partial^3 f}{\partial t^2\partial x} + \delta t\frac{\delta x^2}{2}\frac{\partial^3 f}{\partial t\partial x^2} + \frac{\delta x^3}{6}\frac{\partial^3 f}{\partial x^3}$$

Where,

$$c_0 + c_1\varepsilon_x + c_2\,\varepsilon_x^2 + c_3\,\varepsilon_x^3 = 0$$

$$c_0 = 2\pi\varepsilon_t\varepsilon - 2\pi^2\varepsilon_t^2 + \frac{8\pi^3}{6}\varepsilon_t^3$$

$$c_1 = 2\pi\varepsilon + 4\pi^2\varepsilon_t + \frac{8\pi^3}{2}\varepsilon_t^2$$

$$c_2 = -2\pi^2 + \frac{8\pi^3}{2}\varepsilon_t$$

$$c_3 = \frac{8\pi^3}{6}$$

The solution to this equation can be determined using the Newton-Rhapson iteration method, initiating the process with the root derived from the quadratic equation.

c. Interval coefficient -z ε_z for the function $f(x,z,t) = \cos\sigma t\cos kx\cosh k(h+z)$.

The equation for calculating ε_z is formulated using a method similar to the equations used to calculate ε_t and ε_x, namely by utilizing optimization equations. By applying conditions such as ccos $\sigma t = \sin\sigma t$, $\cos kx = \sin kx$ dan $\cosh k(h+z) = \sinh k(h+z)$, specifically when h is large and z is very small, the elements in the numerator and denominator cancel each other out. Substituting $\delta t = \varepsilon_t L$, $\delta x = \varepsilon_x L$, $\delta z = \varepsilon_z L$, $\sigma = \frac{2\pi}{T}$ and $k = \frac{2\pi}{L}$, we obtain an equation with three variables: ε_t, ε_x and ε_z, with ε_t and ε_x obtained from previous calculations, one variable, and ε_z remains.

The quadratic equation for ε_z is,

$$c_{0,2} + c_{1,2}\varepsilon_z + c_{2,2}\varepsilon_z^2 = 0$$

$$c_{0,2} = (\varepsilon_t + \varepsilon_x)2\pi\varepsilon - 2\pi^2\varepsilon_t^2 + 4\pi^2\varepsilon_t\varepsilon_x - 2\pi^2\varepsilon_x^2$$

$$c_{1,2} = -2\,\pi\varepsilon - 4\pi^2\varepsilon_t - 4\pi^2\varepsilon_x$$

$$c_{2,2} = 2\pi^2$$

This quadratic equation yields two values for ε_z, with the larger value chosen for subsequent analysis. These solutions form the input for a third-degree equation.

$$c_{0,3} = \frac{8\pi^3}{6}{\varepsilon_t}^3 + \frac{8\pi^3}{2}\varepsilon_x{\varepsilon_t}^2 + \frac{8\pi^3}{2}\varepsilon_t{\varepsilon_x}^2 + \frac{8\pi^3}{6}{\varepsilon_x}^3$$

$$c_0 = c_{0,1} + c_{0,2}$$

$$c_{1,3} = -\frac{8\pi^3}{2}{\varepsilon_t}^2 + 8\pi^3\varepsilon_t\varepsilon_x - \frac{8\pi^3}{2}{\varepsilon_x}^2$$

$$c_1 = c_{1,2} + c_{1,3}$$

$$c_{2,3} = -\frac{8\pi^3}{2}\varepsilon_t - \frac{8\pi^3}{2}\varepsilon_x$$

$$c_2 = c_{2,2} + c_{2,3}$$

$$c_3 = \frac{8\pi^3}{6}$$

$c_{0,2}$, $c_{1,2}$ and $c_{2,2}$ are derived from the quadratic equations..

Table (1) below showcases examples of interval coefficient calculation results for various optimization coefficient value ε .

Table 1. The values of interval coefficients

ε	ε_t	ε_x	ε_z
0.005	0.0016	0.00478	0.01452
0.006	0.00192	0.00574	0.01747
0.007	0.00224	0.0067	0.02044
0.008	0.00256	0.00766	0.02342
0.009	0.00288	0.00862	0.02641
0.01	0.00321	0.00959	0.02942

As seen in Table 1, as the value of ε increases, the interval coefficient also increases, indicating a greater contribution from higher-order terms. With these interval coefficients, the contribution coefficients of the higher-order terms are calculated.

Table 2: The values of the contribution coefficients.

ε	μ_t	μ_x	μ_z
0.005	-0.000017	-0.000151	0.001388
0.006	-0.000024	-0.000217	0.002009
0.007	-0.000033	-0.000296	0.002750
0.008	-0.000043	-0.000386	0.003611
0.009	-0.000055	-0.000489	0.004596
0.010	-0.000068	-0.000604	0.005705

IV. CONTRIBUTION FROM EVEN DIFFERENTIAL TERMS

In this stage, the Taylor series is calculated with a very small interval size, so the fourth-order terms and higher are very small, while the third-order term has been absorbed by adding its value to the first-order term. Thus, the remaining Taylor series of the second order takes the form,

$$f(x+\delta x, t+\delta t) = f(x,z,t) + (1+\mu_t)\delta t\frac{\partial f}{\partial t} + (1+\mu_x)\delta x\frac{\partial f}{\partial x} + \frac{\delta t^2}{2}\frac{\partial^2 f}{\partial t^2} + \delta t\,\delta x\frac{\partial^2 f}{\partial t\partial x} + \frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2}$$

The simplification of the subsequent expressions is as follows.

$$\alpha_t = 1 + \mu_t$$

$$\alpha_x = 1 + \mu_x$$

Truncated Taylor series becomes

$$f(x+\delta x, t+\delta t) = f(x,z,t) + \alpha_t\delta t\frac{\partial f}{\partial t}$$

$$+\alpha_x \delta x \frac{\partial f}{\partial x} + \frac{\delta t^2}{2}\frac{\partial^2 f}{\partial t^2} + \delta t\,\delta x \frac{\partial^2 f}{\partial t \partial x} + \frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2}$$

This simplification involves the absorption of contributions from even higher-order differential terms through the application of the Central Different Method operation.

4.1. Function $f(x,t)$

Truncated Taylor series for $(x+\delta x, t+\delta t)$

$$f(x+\delta x, t+\delta t) = f(x,z,t) + \alpha_t \delta t \frac{\partial f}{\partial t} + \alpha_x \delta x \frac{\partial f}{\partial x} + \frac{\delta t^2}{2}\frac{\partial^2 f}{\partial t^2} + \delta t\,\delta x \frac{\partial^2 f}{\partial t \partial x} + \frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2}$$

Truncated Taylor series for $(x-\delta x, t-\delta t)$

$$f(x-\delta x, t-\delta t) = f(x,z,t) - \alpha_t \delta t \frac{\partial f}{\partial t} - \alpha_x \delta x \frac{\partial f}{\partial x} + \frac{\delta t^2}{2}\frac{\partial^2 f}{\partial t^2} + \delta t\,\delta x \frac{\partial^2 f}{\partial t \partial x} + \frac{\delta x^2}{2}\frac{\partial^2 f}{\partial x^2}$$

These two equations are subtracted from each other,

$$f(x+\delta x, t+\delta t) - f(x-\delta x, t-\delta t) = 2\alpha_t \delta t \frac{\partial f}{\partial t} + 2\alpha_x \delta x \frac{\partial f}{\partial x}$$

During this operation, the even higher-order differential terms automatically vanish. $\frac{\partial f}{\partial t}$ will be absorbed at $\frac{\partial f}{\partial x}$, and the last equation is divided by $2\delta x$,

$$\frac{f(x+\delta x, t+\delta t) - f(x-\delta x, t-\delta t)}{2\delta x} = \alpha_t \frac{\delta t}{\delta x}\frac{\partial f}{\partial t} + \alpha_x \frac{\partial f}{\partial x}$$

For very small values of δt and δx, this equation becomes a total spatial differential,

$$\frac{Df}{dx} = \alpha_t \frac{\delta t}{\delta x}\frac{\partial f}{\partial t} + \alpha_x \frac{\partial f}{\partial x}$$

The Taylor series, which has eliminated both even and odd-order differential terms, is,

$$f(x+\delta x, t+\delta t) = f(x,z,t) + \alpha_t \delta t \frac{\partial f}{\partial t} + \alpha_t \delta x \frac{\partial f}{\partial x}$$

Substituting the total spatial differential into $\frac{\partial f}{\partial x}$,

$$f(x+\delta x, t+\delta t) = f(x,t) + \alpha_t \delta t \frac{\partial f}{\partial t} + \alpha_t \delta x \left(\alpha_t \frac{\delta t}{\delta x}\frac{\partial f}{\partial t} + \alpha_x \delta x \frac{\partial f}{\partial x}\right)$$

$$f(x+\delta x, t+\delta t) = f(x,t) + (1+\alpha_x)\alpha_t \delta t \frac{\partial f}{\partial t} + \alpha_x^2 \delta x \frac{\partial f}{\partial x}$$

This equation is written as,

$$f(x+\delta x, t+\delta t) = f(x,t) + \gamma_{t,2} \delta t \frac{\partial f}{\partial t} + \gamma_x \delta x \frac{\partial f}{\partial x}$$

This equation represents a weighted Taylor series for the function$f = f(x,t)$ with weighting coefficients,

$\gamma_{t,2} = (1+\alpha_x)\alpha_t$

$\gamma_x = \alpha_x^2$

The use of the index 2 in $\gamma_{t,2}$ indicates that the weighting coefficient is for a function with two variables, $f = f(x,t)$. The cross-contribution, i.e., the contribution of spatial derivative to the time derivative, is present in $\alpha_x \alpha_t$. In cases where contribution coefficients are negligible, where μ_t and μ_x are both zero, resulting in $\alpha_t = \alpha_x = 1$, we obtain $\gamma_{t,2} = 2$ and $\gamma_x = 1$, these values serve as the fundamental weighting coefficients for the function $f(x,t)$.

4.2. Function with three variables $f = f(x,z,t)$.

To obtain the contribution of even-order differential terms for a function with three variable $f = f(x,z,t)$, the total spatial derivative $\frac{Df}{dx}$ from the function with two variables $f = f(x,t)$ and $\frac{Df}{dz}$ on the function $f = f(z,t)$ where the formulation of $\frac{Df}{dz}$ is performed in the same method as the formulation of $\frac{Df}{dx}$, obtaining

$$\frac{Df}{dz} = \alpha_t \frac{\delta t}{\delta x}\frac{\partial f}{\partial t} + \alpha_z \frac{\partial f}{\partial z}$$

$\frac{Df}{dx}$ and $\frac{Df}{dz}$ is substituted into $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial z}$ on the Taylor Series $f = f(x,z,t)$ which has eliminated its higher-order terms, we obtain

$$f(x+\delta x, z+\delta z, t+\delta t) = f(x,z,t) + \gamma_{t,3}\delta t \frac{\partial f}{\partial t} + \gamma_x \delta x \frac{\partial f}{\partial x} + \gamma_z \delta z \frac{\partial f}{\partial z}$$

$\gamma_{t,3} = (1+\alpha_x+\alpha_z)\alpha_t$

$\gamma_x = \alpha_x^2$

$\gamma_z = \alpha_z^2$

It is observed that the value of γ_x on $f(x,z,t)$ is the same as the value of γ_x for the function $f(x,t)$. In cases where contribution coefficients are neglected, where $\mu_t = \mu_x = \mu_z = 0$ resulting in $\alpha_t = \alpha_x = \alpha_z = 1$, we obtain $\gamma_{t,3} = 3$, while $\gamma_x = \gamma_z = 1$. These values serve as the fundamental weighting coefficients for the function $f(x,z,t)$.

The values of weighting coefficients for various values of ε are presented in Table (3).

Table 3:. The values of weighting coefficients.

ε	$\gamma_{t,2}$	$\gamma_{t,3}$	γ_x	γ_z
0.005	1.99990	3.00237	0.9994	1.00556
0.006	1.99986	3.00344	0.99913	1.00805
0.007	1.99980	3.00471	0.99882	1.01103
0.008	1.99974	3.00619	0.99846	1.01450

0.009	1.99967	3.00789	0.99804	1.01847
0.01	1.99960	3.00980	0.99758	1.02295

It is evident that the values of the weighting functions are not significantly different from the fundamental values, namely $\gamma_{t,2} = 2.0$, $\gamma_{t,3} = 3.0$, $\gamma_x = 1.0$ and $\gamma_z = 1.0$. However, to satisfy a balance equation, it is advisable to use accurate values for the weighting coefficients.

As ε increases, the values of $\gamma_{t,2}$ and γ_x, decrease, while $\gamma_{t,3}$ and γ_z increase. Considering that higher-order terms contain information about the function's characteristics, it is preferable to choose ε such that the coefficients. To obtain the optimal ε coefficient, precise calibration of the model results formulated using the weighted Taylor series is necessary. In the modeling of water wave transformation, this calibration can be applied to evaluate the breaker height or breaker depth generated by the model.

V. THE APPLICATION ON WATER WAVE MODELING

The formulation of various equations in this paper is not provided in detail; only the final results are presented. The research focus is on the weighting coefficients.

5.1. Weighted Laplace Equation and its Solution

a. Weighted Laplace Equation

The continuity equation formulated using the weighted Taylor series takes the form of a weighted continuity equation,

$$\gamma_x \frac{\partial u}{\partial x} + \gamma_z \frac{\partial w}{\partial z} = 0 \qquad \text{……..(3)}$$

$u(x,z,t)$ is the horizontal water particle velocity, and $w(x,z,t)$ is the vertical water particle velocity. The weighted continuity equation cannot be expressed as

$$\frac{\gamma_x}{\gamma_z}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Nor

$$\frac{\partial u}{\partial x} + \frac{\gamma_z}{\gamma_x}\frac{\partial w}{\partial z} = 0$$

since it yields different solutions from (3). Therefore, it can be said that γ_x s directly related to $\frac{\partial u}{\partial x}$ and γ_z is directly related to $\frac{\partial w}{\partial z}$.

Substituting the velocity potential properties, $u = -\frac{\partial \phi}{\partial x}$ and $w = -\frac{\partial \phi}{\partial z}$, into the weighted continuity equation, the weighted Laplace equation is obtained,

$$\gamma_x \frac{\partial^2 \phi}{\partial x^2} + \gamma_z \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{……….(4)}$$

This equation is referred to as the weighted Laplace equation.

b. Solution of the Weighted Laplace Equation

The solution of (4), obtained using the separation of variable method and working with the kinematic bottom boundary condition for a sloping bottom, results in the velocity potential equation,

$$\phi(x,z,t) = G\beta(z)\cos k_x x \sin \sigma t + G\beta(z)\sin k_x x \sin \sigma t$$

In this equation, there are two wave numbers: the wave number in the horizontal axis\

$k_x = \frac{k}{\sqrt{\gamma_x}}$, the wavelength in the horizontal axis $L_x = \frac{2\pi}{k_x}$ and the wave number in the vertical axis $k_z = \frac{k}{\sqrt{\gamma_z}}$, the wavelength in the vertical axis $L_z = \frac{2\pi}{k_z}$.

$$\beta(z) = \frac{\alpha e^{k_z(z+h)} + e^{-k_z(z+h)}}{2}$$

$$\beta_1(z) = \frac{\alpha e^{k_z(z+h)} - e^{-k_z(z+h)}}{2}$$

$$\alpha = \frac{\frac{1}{\sqrt{\gamma_z}} + \frac{1}{\sqrt{\gamma_x}}\frac{dh}{dx}}{\frac{1}{\sqrt{\gamma_z}} - \frac{1}{\sqrt{\gamma_x}}\frac{dh}{dx}}$$

$\frac{dh}{dx}$ is the bottom slope, which is negative for waves moving from deep water to shallow water..

On $\alpha = 1$, $\beta(z) = \cosh k_z(h+z)$; $\beta_1(z) = \sinh k_z(h+z)$

c. Equation for G

The equation for G is obtained by integrating the weighted Kinematic Free Surface Boundary Condition with respect to time (Hutahaean (2023b),

$$G = \frac{\sigma \gamma_{t,2} A}{2k\left(\frac{1}{\sqrt{\gamma_z}} - \frac{kA}{2}\right)\beta(\theta\pi)}$$

$$\beta(\theta\pi) = \frac{\alpha e^{\theta\pi} + e^{-\theta\pi}}{2}$$

θ is the deep water coefficient, where $\frac{\beta_1(\theta\pi)}{\beta(\theta\pi)} = 1$. In this research, $\theta = 1.95$. In the analysis of shoaling-breaking, this coefficient plays a role in determining the breaker depth. A larger θ corresponds to a deeper breaker depth, while a smaller θ results in a shallower breaker depth.

d. Wave Amplitude Function

The equation for G can be expressed as the wave amplitude function,

$$A = \frac{2Gk}{\sigma\gamma_{t,2}}\beta(\theta\pi)\left(\frac{1}{\sqrt{\gamma_z}} - \frac{kA}{2}\right)$$

d. Water Surface Elevation Equation

$\eta(x,t) = A\cos k_x x \cos\sigma t$

e. Wave Number Conservation Equation

$$\frac{dk\left(h + \frac{A}{2}\right)}{dx} = 0$$

f. Energy Conservation Equation

$$G\frac{\partial k}{\partial x} + 2k\frac{\partial G}{\partial x} = 0$$

5.2. Dispersion Equation

By using the weighted Taylor series, and by using the conditions in the formulation of the continuity equation where in a control volume the horizontal velocity only changes on the horizontal axis and the vertical velocity only changes on the vertical axis, and by working on Newton's principle of conservation of momentum we obtain two equations momentum, namely the momentum equation in the horizontal direction and the momentum equation in the vertical direction, respectively, are,

$$\gamma_{t,3}\frac{\partial u}{\partial t} + \gamma_x u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\gamma_{t,3}\frac{\partial w}{\partial t} + \gamma_{z,3} w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

The vertical momentum equation is expressed as the integration of the pressure equation $\frac{\partial p}{\partial z}$, over the water depth. By employing surface dynamic boundary conditions $p_\eta = 0$, the pressure equation p is obtained. Subsequently, differentiating the pressure equation with respect to the horizontal axis and substituting it into the horizontal momentum equation, we apply this at $z = \eta$ to yield the surface momentum equation.

$$\gamma_{t,3}\frac{\partial u_\eta}{\partial t} + \frac{\gamma_x}{2}\frac{\partial u_\eta u_\eta}{\partial x} = -g\frac{\partial \eta}{\partial x}$$

The dispersion equation is obtained by utilizing the surface momentum equation while disregarding the convective acceleration term,

$$\gamma_{t,3}\frac{\partial u_\eta}{\partial t} = -g\frac{\partial \eta}{\partial x}$$

Substituting the velocity potential by employing its inherent property and substituting the water surface equation and the wave amplitude function, the dispersion equation is derived,,

$$\frac{gA}{2}k^2 - \frac{g}{\sqrt{\gamma_z}}k + \gamma_{t,2}\gamma_{t,3}\sigma^2 = 0$$

This dispersion equation is solely for the calculation of wave numbers in deep water. To determine wave numbers in shallow water, the shoaling-breaking model is utilized..

5.3. Shoaling Breaking Equations

By working on the conservation equations, the G dan wave amplitude function (Hutahaean (…)), he shoaling-breaking equations are derived. For waves transitioning from x to $x + \delta x$,

$$\frac{\partial k}{\partial x} = -\frac{4k}{(4h + 3A)}\frac{dh}{dx}$$

$$k_{x+\delta x} = k_x + \delta x\frac{\partial k}{\partial x}$$

$$\frac{\partial A}{\partial x} = \frac{G}{\sigma\gamma_{t,2}}\frac{\partial k}{\partial x}\left(\frac{1}{\sqrt{\gamma_z}} - \frac{kA}{2}\right)\beta(\theta\pi)$$

$$A_{x+\delta x} = A_x + \delta x\frac{\partial A}{\partial x}$$

$$G_{x+\delta x} = e^{\ln G_x - \frac{1}{2}(\ln k_{x+\delta x} - \ln k_x)}$$

5.4. Refraction-Diffraction Equations

The shoaling-breaking equations can be transformed into refraction-diffraction equations using the procedure outlined by Hutahaean (2023a).

VI. OUTCOMES OF THE MODEL

a. The Results of Dispersion Equations

In the following section, the computed wavelengths for waves with a period of 8 seconds are presented. The wave amplitude A varies, with the wave height $H = 2A$. An optimization coefficient of $\varepsilon = 0.01$ where $\gamma_{t,2} = 1.99960$, $\gamma_{t,3} = 3.00980$, $\gamma_x = 0.99758$, $\gamma_z = 1.02295$.. The calculated results are summarized in Table (4) as follows.

Table 4: Wavelength from the modelling outcome

H (m)	L (m)	L_x (m)	L_z (m)	$\frac{H}{L_x}$
2	12.132	12.117	12.264	0.165
2.1	11.784	11.77	11.912	0.178
2.2	11.398	11.384	11.523	0.193
2.3	10.959	10.946	11.078	0.21
2.4	10.434	10.422	10.548	0.23
2.5	9.739	9.727	9.845	0.257

The calculation results for the wavelengths reveal the presence of two wavelengths: the horizontal wavelength L_x

and the vertical wavelength L_z, both exhibiting small differences.

Furthermore, it is observed that within one wave period, the wavelength decreases as the wave height increases, exhibiting different wave steepness. The critical wave steepness according to Michell (1893) is 0.142. The critical wave steepness according to Toffoli, A., Babanin, A., Onaroto, M., and Wased, T. (2010) is 0.170, potentially reaching 0.200. The model results closely align with the critical wave steepness from Toffoli et al., specifically at wave heights of 2.1-2.3 m.

In conclusion, the critical wave steepness from Michell (1893) and Toffoli et al. (2010) indicates the maximum wave height for a given wave period under undisturbed conditions, specifically in deep water. Referring to Toffoli et al.'s criteria, the maximum wave height for waves with a period of 8.0 sec. is expected to be 2.30 m. However, in this study, the maximum wave height for a wave period of 8.0 sec. is found to be 2.50 m, with a wave steepness of 0.257.

The relationship between wave period and wave height, according to Wiegel (1949, 1964), is expressed as,

$$H = \frac{gT^2}{15.6^2}$$

For waves with a period of 8.0 sec., the calculated value is $H = 2.58\ m$.. In this case, the model results closely match the Wiegel (1949, 1964) equation with $\varepsilon = 0.01$.

b. The outcomes of shoaling breaking modelx

The shoaling-breaking model is applied to waves with an amplitude $A = 1.20$ m in coastal waters with a bottom slope $\frac{dh}{dx} = -0.005$, considering various values of ε and a deep water coefficient $\theta = 1.95$.. The wave period is calculated using the Wiegel (1949, 1964) equation:

$$T = 15.6\sqrt{\frac{H}{g}} \quad (sec)$$

Given a wave amplitude $A = 1.20$ m, $H = 2.40\ m$, the $T = 7.716$ sec is obtained.

According to Komar, Paul D., and Gaughan, Michael K. (1968), the breaker height is

$$H_b = 0.39\, g^{1/5}(TH_0)^{2/5} \text{ m.}$$

H_0 is the deep water wave height. For waves with $H_0 = 2.40\ m$, $T = 7.716$ sec the Komar-Gaughan equation yields $H_b = 2.809\ m$. In Table (6), on $\varepsilon = 0.01$, $H_b = 2.812\ m$.

Table 6:Breaking conditions for various values ε

ε	H_b (m)	h_b (m)	$L_{x,b}$ (m)	$\frac{H_b}{L_{x,b}}$	$\frac{H_b}{h_b}$
0.005	2.883	3.619	4.539	0.635	0.797
0.006	2.876	3.615	4.532	0.635	0.796
0.007	2.866	3.607	4.521	0.634	0.794
0.008	2.854	3.6	4.509	0.633	0.793
0.009	2.837	3.588	4.491	0.632	0.791
0.01	2.812	3.568	4.46	0.631	0.788

The results of the breaking conditions for various values of ε are presented in Table (6). From $\varepsilon = 0.005$ to $\varepsilon = 0.01$, the breaking conditions vary only in the third decimal place, with $\delta\varepsilon$ changing by 0.001. However, the difference between the breaking conditions with $\varepsilon = 0.005$ and $\varepsilon = 0.01$ is quite significant.

The breaker depth index from the model is $\frac{H_b}{h_b} = 0.788$, which closely aligns with Mc. Cowan's (1894) criterion of $\frac{H_b}{h_b} = 0.78$. However, the most influential factors in determining the breaker depth and breaker depth index are the deep water coefficient θ, the larger θ, the smaller the breaker depth index. The contribution coefficient only corrects the third decimal place. In Table (6), the breaker depth index is obtained using $\theta = 1.95$.

The smallest breaker steepness $\frac{H_b}{L_{x,b}}$ is 0.631, achieved at $\varepsilon = 0.01$. This breaker steepness is significantly larger than the critical wave steepness from both Michell (1893) and Toffoli et al. (2010). This is because the breaker steepness occurs in regions of high wave energy concentration.

Breaking occurs when $\left(\frac{1}{\sqrt{\gamma_z}} - \frac{kA}{2}\right) = 0$,

$$\frac{kA}{2} = \frac{1}{\sqrt{\gamma_z}}$$

Considering $k_x = \frac{k}{\sqrt{\gamma_x}}$ or $k = k_x\sqrt{\gamma_x}$ and $k_x = \frac{2\pi}{L_x}$ the following is obtained,

$$\frac{H_b}{L_{x,b}} = \frac{2}{\pi\sqrt{\gamma_x}\sqrt{\gamma_z}}$$

Hence, the breaking characteristics are determined by the coefficients of the truncated Taylor series. The smaller γ_x and γ_z then the higher the critical wave steepness $\frac{H_b}{L_{x,b}}$, making breaking more challenging. Conversely, the larger γ_x and γ_z the smaller the critical wave steepness $\frac{H_b}{L_{x,b}}$, making waves more prone to breaking.

c. Results of Refraction-Diffraction Model

The equations derived from the shoaling-breaking equations can be transformed into refraction-diffraction equations using the method proposed by Hutahaean (2003). The refraction-diffraction model is executed with two ε values: 0.005 and 0.01, considering the bathymetry of a submerged island (Fig (1)). Contour plots of 2D wave height are presented in Fig (2), with (a) using $\varepsilon = 0.005$

and (b) using $\varepsilon = 0.01$. Three-dimensional contour plots are depicted in Fig (3).

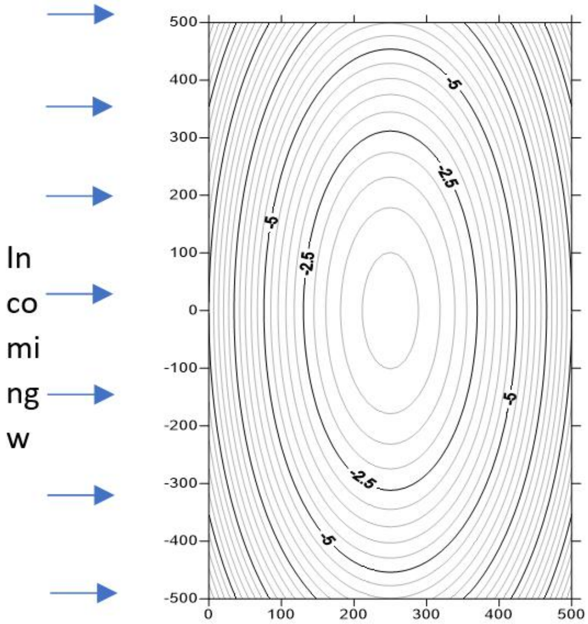
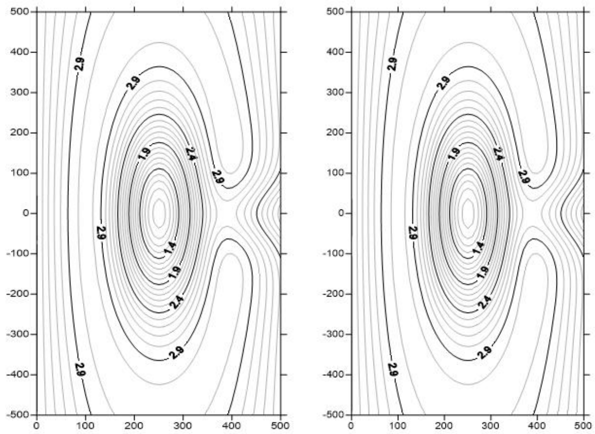


Fig.1: Contour of Submerged Island Bathymetry



(a) (b)

Fig.2: Contour of Wave Height

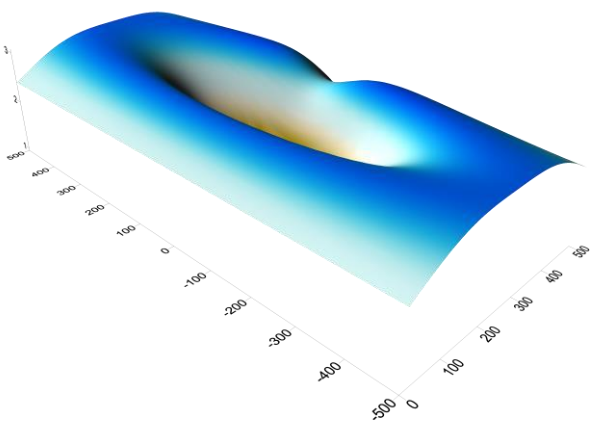


Fig.3: 3D Contour of Wave Height

Both refraction-diffraction model results show no discernible differences. From these findings, it can be concluded that the contribution coefficient does not play a significant role; the primary factors at play are the main values of weighting coefficients.

In conclusion, the results of the study on weighting coefficients are as follows:
The baseline values of weighting coefficients are,
For the function $f(x,t)$: $\gamma_{t,2} = 2.0$, $\gamma_x = 1.0$
For the function $f(x,z,t)$: $\gamma_{t,3} = 3.0$, $\gamma_x = 1.0$, $\gamma_z = 1.0$
The recommended corrected values for the weighting coefficients are based on the optimization coefficient $\varepsilon = 0.01$ where, $\gamma_{t,2} = 1.99960$, $\gamma_{t,3} = 3.00980$, $\gamma_x = 0.99758$, $\gamma_z = 1.02295$.

VII. CONCLUSION

As widely recognized, the precision of the Taylor series hinges on the inclusion of high-order terms. The greater the number of these terms, the more accurate the approximation becomes. High-order terms encapsulate essential features of the function under consideration in the series, making them indispensable. When restricting a series to only first-order terms, the incorporation of a set of weighting coefficients becomes necessary to account for the high-order terms, resulting in a weighted Taylor series.

The Central Difference Method establishes the foundational values for these weighting coefficients. These baseline values undergo refinement through the inclusion of contribution coefficients derived from high-order terms with odd differentials, introducing relatively minor adjustments. In the context of modeling shoaling breaking, these contribution coefficients make subtle corrections to breaking characteristics, particularly at the third decimal place. Notably, the baseline values of weighting coefficients are derived without a prerequisite knowledge of the functional form, while determination of contribution coefficients relies on knowledge of the functional form. Thus, it is evident that baseline values hold a general applicability across various functional forms.

Despite the modest impact of contribution coefficients on both weighting coefficients and model outcomes, their significance should not be understated, as they play a crucial role in ensuring accuracy in balance equations. In models demanding high numerical precision, such as time series models, it is advisable to utilize weighting coefficients corrected by contribution coefficients to enhance accuracy.

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