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Utility of Laplace Transform in Mathematics

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Abstract*—*The paper seeks to analyze the use of Laplace transform in mathematics. However it contributes in mathematics as well as in arena of physics and engineering also. Laplace transform is an important skill to solve linear ordinary and partial differential equations with constant coefficients under suitable initial and boundary conditions. It is a good technique to simplify complex differential equations to a simpler form having polynomials in the area of stability and control. The current far-reaching use of the transform (mainly in engineering) happened during and soon after 2nd World War ,With the ease of application of Laplace transforms in myriad of scientific applications, many research softwares have made it possible to activate the Laplace transformable equations directly supporting the researchers. The transformation is usually used in stochastic performance modelling and analysis of computer and communication systems. It gets significant applications in various areas of physics, electrical engineering, control engineering, optics, mathematics and signal processing.

I. INTRODUCTION

In the subject of Mathematics ,a transformation is a way that changes one function into another function. For example, an operation of differentiation on differentiable function is a transformation as it changes a function into another function known as $f'(x)$.The topic of linear transformation reduces a given initial value problem to an algebraic operation and the solution of initial value problem is obtained directly without finding the general solution. Laplace Transformation is a way to solve differential equations. So the knowledge of the subject area is also necessary for science and engineering background scholars for their work. The important field of Mathematical Analysis is Laplace transformation is referred as integral transforms inserting applications in varigated fields like **engineering technology, basic sciences, mathematics and in economics**. It is also used to find the solution of differential equations at boundary value. A great French Mathematician named as **Pierre Simon Marquis De Laplace(1749-1827)** made valuable contributions to potential theory ,astronomy, special functions and probability theory .It would be interesting to know that a British engineer **Oliver Heaviside (1850-1925)** developed the significant Laplace Transforms techniques after many years of death of Laplace and due to that these transforms are called as Heaviside calculus. Many scholars have highlighted the role of Laplace transform in Mathematics as well as other branches of Physics and engineering such as D. Poularikas explained the transforms and applications [2000] in his work, Kapur, J.N.[2005] mentioned Mathematical Modelling in his work, Karan Asher[2013] described an Introduction to Laplace Transform, Lokanath Sahoo [2020] discussed application of laplace transform for solving problems on newton's law of cooling and so on. I also tried to study the role of this transformation in mathematics after study of literature related to this transform. Laplace Transformations are most commonly used in stochastic performance modelling and analysis of computer and communication systems. Laplace transform is applicable in probability theory like first passage times of stochastic processes, Markov chains and renewal theory. In Physics and engineering, it is used for analysis of linear time-invariant systems such as electric circuits, harmonic oscillations, optical devices etc.

Definition :- Laplace Transformation of a function : Let f(t) be a function defined for all $t \geq 0$,Then the Laplace Transformation of a function f(t) ,represented by L(f(t)) is defined as

$$L(f(t)) = \int_0^\infty f(t)e^{-st}dt$$ provided the integral exists

The parameter s is a real or complex number, frequently it is assumed as a positive real number.

It is seen that L(f(t)) is a function of s and represented by F(s) i.e. L(f(t)) = F(s)

We can say as f(t) = L^{-1} (F(s)) and f(t) is named as **inverse Laplace Transform of F(s)**.

Note: The symbol 'L' is called the Laplace Transform operator and when applied on f(t), it converts into F(s).The Laplace transform of f(t) exists if the above integral converges for some value of t, otherwise it does not exists.

Theorem: Sufficient condition for existence theorem on Laplace transform: If f(t) is a piecewise continuous function in every finite interval in its domain $t \geq 0$ and is of exponential order 'a' , then the Laplace transform of f(t) exists for all $s > a$.

Let us have a problem of integral equation (by using laplace transform) $f(t) = t + 2\int_0^t \cos(t-u)f(u)du$

Solution: The given integral equation is

$$f(t) = t + 2\int_0^t \cos(t-u)f(u)du$$

Write it as $f(t) = t + 2\cos t * f(t)$

Taking laplace transform on both sides, we get

$$F(s) = \frac{1}{s^2} + 2.\frac{s}{1+s^2}.F(s)$$

Or $F(s)(1 - \frac{2s}{1+s^2}) = \frac{1}{s^2}$

Or $F(s) = \frac{1+s^2}{s^2(s-1)^2}$

Taking inverse laplace transform on both sides,we get

$$L^{-1}(F(s)) = L^{-1}(\frac{1+s^2}{s^2(s-1)^2}) \quad or \quad f(t) = L^{-1}(\frac{1+s^2}{s^2(s-1)^2})$$

Or $f(t) = L^{-1}(\frac{1}{s^2(s-1)^2}) + L^{-1}(\frac{1}{(s-1)^2})$

$$= \int_0^t \int_0^t t e^t \, dtdt + t e^t$$

Or

$$f(t) = t e^t - 2e^t + t + 2 + t e^t = 2t e^t - 2e^t + t + 2$$

We proceed further by taking more examples of different types

Remark: Above condition is only sufficient for existence theorem on Laplace transform. Converse is not true. There may be a function having Laplace transform but may not satisfy the existence condition.

Integral Equation: An equation of the form

$f(t) = h(t) + \int_a^b f(u).g(u,t)du$ is called an integral equation -----------------------(1)

In this equation h(t) and g(u, t) are known, limits a and b are either constants or functions of t. We have to determine the function f(t) under the integral sign.

The equation $f(t) = h(t) + \int_a^b f(u).g(t-u)du$ is also of type (1), therefore it is an integral equation. This equation is called integral equation of convolution type.

The equation $\int_0^t \frac{f(u)}{(t-a)^n}du = g(t)$ where g(t) is also an integral equation called Abel's integral equation.

A. Solution of Linear differential Equations with constant co-efficients by Laplace Transform of derivatives.

Step1. Take Laplace transform of both sides using the method of derivatives,using initial conditions.

Step2. Step1 gives an algebraic equation called subsidiary equation.

Step3. Divide by co-efficient of $\bar{y}$ that is used in place of F(s).

Step4. Have the inverse laplace transform of both sides.

If f(t) is a function and and its derivatives are represented by $f'(t)$, $f''(t)$, -------- etc

We are familiar that **L(f'(t)) = s F(s) – f(0)**

L(f''(t) = s^2 F(s) –s f(0) –f'(0)

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L(f^n(t)) = s^n F(s) –s^{n-1} f(0) –s^{n-2} f'(0)---------- +f^{n-1}(0)

Where **L(f(t)) = F(s)**

We have another exampleby laplace transform methodin Solution of the following differential equation

$$\frac{d^2y}{dt^2}+y=6\cos 2t \quad , \quad y'(0)=1, \; y(0)=3$$

Solution : The given equation is $\frac{d^2y}{dt^2}+y=6\cos 2t$

Having laplace transform on both sides, we obtain

$$(s^2\bar{y}-s\,y(0)-y'(0))+\bar{y}=6\frac{s}{(s^2+4)}$$

Or

$$(s^2\bar{y}-3s-1)+\bar{y}=6\frac{s}{s^2+4} \qquad as\; y(0)=3, y'(0)=1$$

Or $(s^2+1)\bar{y}= \; 1+3s+6\frac{s}{s^2+4}$

Or $\bar{y}=\frac{1}{s^2+1}+3\frac{s}{s^2+1}+6\frac{s}{(s^2+4)(s^2+1)}$

Having **inverse**laplace transform on both sides, we obtain

$$y=L^{-1}\frac{1}{s^2+1}+3L^{-1}(\frac{s}{s^2+1})+6L^{-1}(\frac{s}{(s^2+4)(s^2+1)})$$

$$=\sin t+3\cos t+6\int_0^t\frac{1}{2}\sin 2u\cos(t-u)du$$

(convolution theorem)

$$=\sin t+3\cos t+\frac{3}{2}\int_0^t[\sin(2u+t-u)+\sin(2u-t+u)]du$$

$$=\sin t+3\cos t+\frac{3}{2}\int_0^t[\sin(u+t)+\sin(3u-t)]du$$

$$=\sin t+3\cos t+\frac{3}{2}[-\frac{\cos(u+t)}{1}-\frac{\cos(3u-t)}{3}]_0^t$$

$$=\sin t+3\cos t+\frac{3}{2}[-\frac{\cos(2t)}{1}-\frac{\cos(2t)}{3}+\cos t+\frac{\cos t}{3}]$$

$$=\sin t+3\cos t-2\cos 2t+2\cos t$$

$$=\sin t+5\cos t-2\cos 2t$$

Similarily we can produce examples of

B. Solution of ordinary differential equation with variable co-efficient by transform method .

Here we use $L(y(t))=\bar{y}(s)$,then

$$L(t^n y(t))=(-1)^n\frac{d^n}{ds^n}[\bar{y}(s)]$$

$$i.e.\;\; L(t^n y(t))=(-1)^n\frac{d^n}{ds^n}[L(y(t)]$$

Application of L.T can be seen in solution of the differential equation $t\frac{d^2y}{dt^2}+(t-1)\frac{dy}{dt}-y=0,$ $y(0)=5, y(\infty)=0$

Solution: The given differential equation is

$$t\frac{d^2y}{dt^2}+(t-1)\frac{dy}{dt}-y=0 \;\; ----------(1)$$

Having Laplace transform both sides of equation (1)

$$L(t\frac{d^2y}{dt^2})+L(t\frac{dy}{dt})-L(\frac{dy}{dt})-L(y)=0$$

Or $(-1)^n\frac{d}{ds}(L(\frac{d^2y}{dt^2}))-\frac{d}{ds}(L(\frac{dy}{dt}))-L(\frac{dy}{dt})-\bar{y}=0$

Or $-\frac{d}{ds}[s^2\bar{y}-sy(0)-y'(0)]-\frac{d}{ds}(s\bar{y}-y(0))-(s\bar{y}-y(0))-\bar{y}=0$

Or

$$-\frac{d}{ds}[s^2\bar{y}-5s-0]-\frac{d}{ds}(s\bar{y}-5)-(s\bar{y}-5)-\bar{y}=0$$

Or

$$-[s^2\frac{d\bar{y}}{ds}+2s\bar{y}-5]-s\frac{d\bar{y}}{ds}-\bar{y}+0-(s\bar{y}-5)-\bar{y}=0$$

Or

$$s^2\frac{d\bar{y}}{ds}+2s\bar{y}-5+s\frac{d\bar{y}}{ds}+\bar{y}+0+s\bar{y}-5+\bar{y}=0$$

Or $(s^2+s)\frac{d\bar{y}}{ds}+(3s+2)\bar{y}-10=0$

Or $\frac{d\bar{y}}{ds} + \frac{3s+2}{s(s+1)}\bar{y} = \frac{10}{s(s+1)}$ which is a linear differential equation.

We know that $\frac{dy}{dx} + Py = Q$ **where P , Q are functions of x only,**

The solution is

$$y.\ I.F = \int Q(I.F.)\,dx \text{ where } I.F. = e^{\int P dx}$$

Here I.F.=

$$e^{\int \frac{3s+2}{s(s+2)}ds} = e^{\int (\frac{2}{s}+\frac{1}{s+1})ds} = e^{2\log s + \log(s+1)} = s^2(s+1)$$

Then the solution is

$$\bar{y}(s^2(s+1)) = \int \frac{10}{s(s+1)}.s^2(s+1)\,ds + c$$

Or

$$\bar{y}(s^2(s+1)) = \int 10s\,ds + c = 5s^2 + c \quad -----(2)$$

Taking limits as s →0 on both sides, we obtain c = 0

From(2) , $\bar{y} = \frac{5s^2}{s^2(s+1)} = \frac{5}{s+1)}$

Taking inverse laplace transform both sides, we obtain

$$y = 5e^{-t}$$

Obviously where t→ ∞ ,y→ 0

Hence the required solution is $y = 5e^{-t}$

C. Solution of Simultaneous Linear Equations with constant co-efficient by transform method

We can solve the following simultaneous equations using Laplace transform method

$$\frac{dx}{dt} - y = e^t,\ \frac{dy}{dt} + x = \sin t,\ \ x(0) = 1, y(0) = 0$$

Solution: The equations are given as

$$\frac{dx}{dt} - y = e^t,\ \frac{dy}{dt} + x = \sin t$$

Having laplace transform on both sides,we obtain

$$L(\frac{dx}{dt}) - L(y) = \frac{1}{s-1}$$

$$i.e.\quad s\bar{x} - x(0) - \bar{y} = \frac{1}{s-1}$$

$$and \quad L(\frac{dy}{dt}) - L(x) = \frac{1}{s^2+1}$$

$$i.e.\quad s\bar{y} - y(0) + \bar{x} = \frac{1}{s^2+1}$$

But is given that x(0)=1 , y(0)= 0

$$i.e.\quad s\bar{x} - 1 - \bar{y} = \frac{1}{s-1}$$

$$\Rightarrow\ s\bar{x} - \bar{y} = \frac{1}{s-1} + 1 = \frac{s}{s-1} \quad ------(1)$$

$$and \quad s\bar{y} - 0 + \bar{x} = \frac{1}{s^2+1}$$

$$\Rightarrow\ s\bar{y} + \bar{x} = \frac{1}{s^2+1} \quad -------(2)$$

Multiplying (1) by s ,

$$\Rightarrow\ s^2\bar{x} - s\bar{y} = \frac{s1}{s-1} + s \quad ---------(3)$$

Adding (2) and (3),we have

$$\Rightarrow\ s^2\bar{x} + \bar{x} = \frac{1}{s^2+1} + \frac{s}{s-1} + s$$

$$\Rightarrow\ \bar{x} = \frac{1}{(s^2+1)^2} + \frac{s}{(s-1)(s^2+1)} + \frac{s}{(s^2+1)}$$

Taking inverse Laplace transform, we obtain

$$x = \frac{1}{2}(\sin t - t\cos t) + L^{-1}(\frac{s}{(s-1)(s^2+1)}) + \cos t$$

$$\Rightarrow x = \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}L^{-1}[\frac{1}{(s-1)} - \frac{s-1}{(s^2+1)}] + \cos t$$

$$\Rightarrow x = \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}[L^{-1}(\frac{1}{s-1}) - L^{-1}(\frac{s}{s^2+1}) + L^{-1}(\frac{1}{s^2+1})] + \cos t$$

$$\Rightarrow x = \frac{1}{2}(\sin t - t\cos t) + \frac{1}{2}[e^t - \cos t + \sin t] + \cos t$$

$$\Rightarrow x = \frac{1}{2}[(\sin t - t\cos t) + e^t - \cos t + \sin t + 2\cos t]$$

$$\Rightarrow x = \frac{1}{2}[(\sin t - t\cos t) + e^t + \cos t + \sin t]$$

$$\Rightarrow x = \frac{1}{2}[2\sin t - t\cos t + \cos t + e^t] \text{ ------(4)}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{2}[(2\cos t - \cos t + t\sin t - \sin t + e^t]$$

As $\frac{dx}{dt} - y = e^t$

$$\Rightarrow y = \frac{1}{2}[\cos t + t\sin t - \sin t + e^t] - e^t$$

$$\Rightarrow y = \frac{1}{2}[\cos t + t\sin t - \sin t - e^t] --- (5)$$

Hence we get the Solution from equations (4) and (5)

Laplace transform method is used to solve the following simultaneous equations

$\frac{dx}{dt} = 5x + y$, $\frac{dy}{dt} = x + 5y$ when x(0)= -3 , y(0) = 7

Solution: The given parametric equations are

$$\frac{dx}{dt} = 5x + y \;,\; \frac{dy}{dt} = x + 5y$$

Taking laplace transform of both equations, we obtain

$$s\overline{x} - x(0) = 5\overline{x} + \overline{y}$$

$$s\overline{y} - y(0) = \overline{x} + 5\overline{y}$$

But given that x(0) = -3and y(0) =7

Hence $s\overline{x} + 3 = 5\overline{x} + \overline{y}$

And $s\overline{y} - 7 = \overline{x} + 5\overline{y}$

i.e. $(s-5)\overline{x} - \overline{y} + 3 = 0$

and $\overline{x} + (5-s)\overline{y} + 7 = 0$

On solving,we obtain

$$\frac{\overline{x}}{-7-3(5-s)} = \frac{\overline{y}}{3-7(s-5)} = \frac{1}{-(s-5)^2+1}$$

Therefore

$$\overline{x} = \frac{3s-22}{-(s-5)^2+1} = \frac{3s-22}{-s^2+10s-24} = -\frac{3s-22}{s^2-10s+24}$$

$$= -\frac{(3s-22)}{(s-4)(s-6)}$$

$$\overline{x} = -[\frac{5}{s-4} - \frac{2}{s-6}] = -\frac{5}{s-4} + \frac{2}{s-6}$$

Taking inverse laplace transform ,we get

$x = -5e^{4t} + 2e^{6t}$

Similarly

$$\overline{y} = \frac{-7s+38}{-(s-5)^2+1} = -\frac{-7s+38)}{s^2-10s-24} = -\frac{(-7s+38)}{(s-6)(s-4)}$$

or $\overline{y} = -[-\frac{5}{s-4} - \frac{2}{s-6}]$

or $\overline{y} = \frac{5}{s-4} + \frac{2}{s-6}$

after having inverse laplace transform both sides , we get

$y = 5e^{4t} + 2e^{5t}$

Hence the solution is $x = -5e^{4t} + 2e^{6t}$,

$y = 5e^{4t} + 2e^{5t}$

Laplace Transform in Probability Theory

The Laplace transform is defined as an expected value in pure and applied probability theory. Let X is the random variable with probability density function f (say), then the Laplace transform of f is given as the expectation of:

L{f}(s) = E[e^{-sX}], which is referred to as the Laplace transform of random variable X itself.

Significance of Laplace Transforms:

Usually we listen that Mathematics is the foundation of all sciences. The concept of Laplace Transform is equally important in other fields of study besides mathematics. It's application is much appropriate in arena of science and engineering.

- Engineers make use of above mentioned Transform to solve swiftly differential equations occurring in the analysis of electronic circuits.
- It is applied to simplify calculations in system modeling, where large number of differential equations are used.

- By making use of Laplace Transform, one finds help in solving digital signal processing problems.
- Laplace Transform is used to get the true form of radioactive decay. It makes comfortable to study analytic part of Nuclear physics possible.
- This Transform is much used for process controls. It is supportive to analyze the variables which when changed, produce desired manipulations in the result.

II. METHODOLOGY

I studied many papers and books thoroughly and then reviewed application of laplace transform in the subject of mathematics as well as in other branches of science and engineering.

III. CONCLUSION

In the present paper we have seen how Laplace transform is supportive to solve different problems in the subject of mathematics. The aforementioned topic is favorable in multifarious arena of Probability theory, Physics, Electrical engineering, Control engineering, Economics, Mathematics, Signal processing and Electronics engineering. The scholars can get benefitted by knowledge of above mentioned topic and can do their work in a better way.

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