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ϖ-Interpolative Ciric-Reich-Rus-Type Contactions in m-metric space

Muhammad Umair[1], Ahmad Hassan[2], Ali Hassan[3], Abid Hussain[4], Habiba Faiz[5], Sana Naz Maqbool[6], Muahmmad Jamil[7]

[1,2 *] International Islamic University, Pakistan
[3,4 *] The Islamia University of Bahawalpur, Pakistan
[5,7 *] The university of Lahore, Pakistan
[6 *] The University of Management and Technology, Pakistan

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***Abstract*— In this paper, using the concept of ϖ-admissibility, we prove some fixed -point results for interpolate Ciric-Reich-Rus-type contraction mappings. We also present some consequences and a useful example.**

I. INTRODUCTION AND PRELIMINARIES:

In [2], the notion of an interpolative Kannan-type contraction was introduced and the following fixed-point theorem was stated: A self-mapping T on a complete metric space (X, d) such that:

$$d(Tx, Ty) \leq \lambda[d(x, Tx)]^{\alpha}[d(y, Ty)]^{\alpha-1}$$

where $\lambda \in [0,1)$ and $\alpha \in (0,1)$, and $x, y \in X$ with $x \neq Tx$, has a unique fixed point in X. Very recently, the authors in [2] (see also [3]) pointed out a gap in [2], that is the guaranteed fixed point in the theorem above need not be unique. In 2012, Bessem Samet [5] introduced a new concept of contraction named α-admissible and proved a fixed-point theorem which generalizes Banach contraction principle. Bessem Samet further established fxed point result for $\alpha - \psi$ −contractive type mappings and establish various fixed-point theorems for such mappings in complete metric spaces. Afterwards Karapinar and Samet [5] generalized these notions to obtain fixed point results. The aim of this paper is to modify further the notions of $\alpha - \psi$ −contractive and α −admissible mappings and establish fixed point theorems for such mappings in complete metric spaces. The notion of α-admissible mapping is an interesting increase in improving the Banach contraction mapping in order to make the mapping become more general including the case that it is continuous or discontinuous. There is now extensive variety of literature dealing with fixed point problems via α −admissible mappings. In 2014, M. Asadi et al. [6] extended the concept of partial metric space to an m −metric space, and showed that their definition is a real generalization of partial metric by presenting some examples. Partial metric space (in short PMS), is one of the attempts to generalize the notion of the metric space that by replacing the condition $d(x, x) = 0$ with the condition $d(x, x) \leq d(x, y)$ for all $x, y \in X$ in the definition of the metric [7, 8]. Regarding the contributions of these authors, we shall call the following result the Ciri´c-Reich-Rus theorem, by which our main result is inspired. Ciri´c-Reich-Rus theorem: A self-mapping ´ T on a complete metric space (X, d) such that:

$$d(Tx, Ty) \leq \lambda[d(x, y) + d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, where $\lambda \in \left(0, \frac{1}{3}\right)$, possesses a unique fixed point. Denote by Ψ the set of all nondecreasing self-mappings ψ on $[0, \infty)$ such that:

$$\sum_{n=1}^{\infty} \psi^{n}(t) < \infty \quad for\ each \quad t > 0$$

Note that for $\psi \in \Psi$, we have $\psi(0) = 0$ and $\psi(t) < t$ for each $t > 0$; see, e.g., [9,10]. The notion of ϖ −orbital admissible maps was introduced by Popescu as a refinement of the concept of α-admissible maps of Samet et al.[11].

Definition [14] Let $\varpi: X \times X \to [0, \infty)$ be a mapping and $X \neq \emptyset$. A self mapping $T: X \to X$ is said to be an ϖ −orbital admissible if for all $s \in X$, we have: $\varpi(s, Ts) \geq 1 \Rightarrow \varpi(Ts, T^2 s) \geq 1$

Definition [6] For a given non empty set X, we say that a function $m: X \times X \to [0,1)$ is an m-metric if

(m1) $m(x, x) = m(y, y) = m(x, y) \Leftrightarrow x = y$,

(m2) $m_{x,y} \leq m(x, y), where\ m_{x,y} := min\{m(x, x), m(y, y)\}$,

(m3) $m(x, y) = m(y, x)$,

(m4) $\left(m(x, y) - m_{x,y}\right) \leq \left(m(x, z) - m_{x,z}\right) + \left(m(z, y) - m_{z,y}\right)$.

In this case, the pair (x, m) is called an m −metric space.

Example [6] Let $X = [0, +\infty)$ and $m(x, y) = x^2 + y^2$ on X. Then (X, m) is an m −metric space.

Solution

(m1) If $x = y$ then $m(x, x) = m(x, y) = m(y, y)$ where

$$m(x, x) = x^2 + x^2 = 2x^2,$$
$$m(x, y) = x^2 + y^2 = 2x^2\ because\ x = y,$$
$$m(y, y) = y^2 + y^2 = 2y^2 = 2x^2.$$

Conversely suppose that $m(x, x) = m(x, y) = m(y, y)$ then $x = y$.

(m2) We have to show that $m_{xy} \leq m(x, y)$ where $m_{xy} = min\{m(x, x), m(y, y)\}$,

If $x < y$ then

$$m_{x,y} = min\{m(x, x), m(y, y)\} = m(x, x)$$

Similarly for $x > y \Rightarrow m_{x,y} \leq m(x, y)$

(m3) If

$$m(x, y) = x^2 + y^2 = y^2 + x^2 = m(y, x) \Rightarrow m(x, y) = m(y, x)$$

(m4) We have to show that

$$m(x, y) - m_{x,y} \leq \left(m(x, z) - m_{x,z}\right) - \left(m(z, y) - m_{z,y}\right)$$

Take $x = 1, y = 3$ and $z = 5$ where

$$m(1,3) = min\{m(1,1), m(3,3)\} = min\{1,9\} = 1,$$
$$m(1,5) = min\{m(1,1), m(5,5)\} = min\{1,25\} = 1,$$
$$m(3,5) = min\{m(3,3), m(5,5)\} = min\{9,25\} = 9.$$

Then

$$m(1,3) - m_{1,3} \leq \left(m(1,5) - m_{1,5}\right) + \left(m(5,3) - m_{5,3}\right),$$
$$10 - 1 \leq (26 - 1) + (36 - 9),$$
$$9 \leq 25 + 27 = 52.$$

Similarly for all $x, y \in X$ which satisfied

$$m(x, y) - m_{x,y} \leq \left(m(x, z) - m_{x,z}\right) - \left(m(z, y) - m_{z,y}\right).$$

So all condition of m −metric space are satisfied so (x, m) is a m −metric space on X.

Example [6] Let m be an m −metric space. Put

(1) $m^z(x, y) = m(x, y) - 2m_{xy} + M_{xy}$,

(2) $m^s(x, y) = m(x, y) - m_{xy}\ if\ x \neq y, and\ m^s(x, y) = 0\ if\ x = y.$

Then m^z and m^s are ordinary metrics.

As mentioned in [2], each m −metric on X generates a T_o topoplogy τ_m on X. Then set

$$\{B_m(x, \varepsilon): x \in X, \varepsilon > 0\}$$

where

$$B_m(x, \varepsilon) : \{y \in X, m(x, y) < m_{x,y} + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$, forms a basis of τ_m.

Remark [6] For every $x, y \in X$

1. $0 \leq M_{xy} + m_{xy} = m(x, x) + m(y, y)$,

2. $0 \leq M_{xy} - m_{xy} = |m(x, x) + m(y, y)|$,

3. $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + \left(M_{zy} + m_{zy}\right)$

Definition [6] Let (X, m) be a m-metric space. Then:

1. A sequence $\{x_n\}$ in m-metric space (X, m) converges to a point $x \in X$ if and only if

$$lim_{n \to \infty}(m(x_n, x) - m_{x_n, x}) = 0$$

2. A sequence $\{x_n\}$ in m-metric space (X, m) is called an m −Cauchy sequence if

$$lim_{n \to \infty}\left(m(x_n, x_m) - m_{x_n, x_m}\right), \quad lim_{n \to \infty}\left(M_{x_n, x_m} - m_{x_n, x_m}\right),$$

exist (and are finite).

3. An m −metric space (X, m) is said to be complete if every m −Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_m, to a point $x \in X$ such that

$$lim_{n\to\infty}\left(m(x_n, x) - m_{x_n,x}\right) = 0 \quad and \quad lim_{n\to\infty}(M_{x_n,x} - m_{x_n,x}) = 0$$

Lemma 1: If $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ are two sequence such that $x_n \to \infty$ and $y_n \to y$ as $n \to \infty$ in an m-metric space (X, m), then

$$lim_{n\to\infty}(m(x_n, y_n) - m_{x_n,y_n}) = m(x, y) - m_{x,y}$$

Lemma 2: If $\{x_n\}_{n\in N}$ be an sequence such that $x_n \to \infty$ as $n \to \infty$ in an m −metric space (X, m), then

$$lim_{n\to\infty}(m(x_n, y) - m_{x_n,y}) = m(x, y) - m_{x,y}$$

(H) If $\{u_n\}$ is a sequence in X such that $\varpi(u_n, u_{n+1}) \geq 1$ for each n and $u_n \to u \in X$ as $n \to \infty$, then there exists $\{u_{n(k)}\}$ from $\{u_n\}$ such that $\varpi(u_{n(k)}, u) \geq 1$ for each k.

Main Body:

At the start of this section, we define a ϖ −interolative Ciric-Reich-Rus-type contraction in m-metric space, which is cited in a well-known paper on Ciric-Reich-Rus-type contraction metric by Karapinar [1].

First, we initiate the concept of ϖ −interolative Ciric-Reich-Rus-type contraction.

Definition-1: Let (X, m) be a m-metric space. The mapping $T: X \to X$ is said to be an ϖ −interolative Ciric-Reich-Rus-type contraction if there exist $\psi \in \Psi, \varpi : X \times X \to [0, \infty)$ and positive reals $\gamma, \beta > 0$, verifying $\gamma + \beta < 1$, such that:

$$\varpi(u, v)m(Tu, Tv) \leq \psi\left([m(u, v)]^\beta . [m(u, Tu)]^\gamma . [m(v, Tv)]^{1-\gamma-\beta}\right) \quad (2.1)$$

for all $u, v \in X \backslash Fix(T)$, where $Fix(t)$ denotes the set of fixed point of T (that is, point $a \in X$ such that $Ta = a$).

The essential main result is given as follows.

* We use the m-metric space instead of metric in [1] theorem 1.

Theorem 1: Suppose a continuous self-mapping $T : X \to X$ is ϖ −orbital admissible and forms an ϖ −interpolative Ciric-Reich-Rus-type contraction on a complete m-metric space (X, m). If there exist $u_0 \in X$ such that $\varpi(u_0, Tu_0) \geq 1$, then T possesses a fixed point in X.

Proof Let $u_0 \in X$ be a point such that $\varpi(u_0, Tu_0) \geq 1$. Let $\{u_n\}$ be a sequence defined by $u_n = T^n(u_0)$, $n > 0$. If for some n_0, we have $u_{n_0} = u_{n_0+1}$, then u_{n_0} is a fixed point of T, which ends the proof. Otherwise, $u_n \neq u_{n+1}$ for each $n \geq 0$. we have $\varpi(u_0, u_1) > 1$. Since T is ϖ −orbital admissible,

$$\varpi(u_1, u_2) = \varpi(Tu_0, Tu_1) \geq 1.$$

Continuing as above, we obtain that:

$$\varpi(u_n, u_{n+1}) \geq 1 \; for \; all \; n \geq 0. \quad (2.2)$$

Taking $u = u_n$ and $v = u_{n-1}$ in (2.1), we find that:

$$m(u_{n+1}, u_n) \leq \varpi(u_n, u_{n-1})m(Tu_n, Tu_{n-1})$$
$$\leq \psi\left([m(u_n, u_{n-1})]^\beta . [m(u_n, Tu_n)]^\gamma . [m(u_{n-1}, Tu_{n-1})]^{1-\gamma-\beta}\right)$$
$$\leq \psi\left([m(u_n, u_{n-1})]^\beta . [m(u_n, u_{n+1})]^\gamma . [m(u_{n-1}, u_n)]^{1-\gamma-\beta}\right)$$
$$\leq \psi([m(u_{n-1}, u_n)]^{1-\gamma} . [m(u_n, u_{n+1})]^\gamma) \quad (2.3)$$

In particular, as $\psi(t) < t$ for each $t > 0$,

$$m(u_{n+1}, u_n) \leq \psi([m(u_{n-1}, u_n)]^{1-\gamma} . [m(u_n, u_{n+1})]^\gamma)$$
$$< [m(u_{n-1}, u_n)]^{1-\gamma} . [m(u_n, u_{n+1})]^\gamma \quad (2.4)$$

We derive:

$$[m(u_n, u_{n+1})]^{1-\gamma} < [m(u_{n-1}, u_n)]^{1-\gamma} \quad (2.5)$$

Therefore:

$$m(u_n, u_{n+1}) < m(u_{n-1}, u_n) \; for \; all \; n \geq 1 \quad (2.6)$$

Hence, the positive sequence $\{m(u_{n-1}, u_n)\}$ is decreasing. Eventually, there is a real $\ell \geq 0$ in order that $lim_{n\to\infty} m(u_{n-1}, u_n) = \ell$. Taking into account (2.6),

$$[m(u_{n-1}, u_n)]^{1-\gamma}[m(u_n, u_{n+1})]^\gamma \leq [m(u_{n-1}, u_n)]^{1-\gamma}[m(u_{n-1}, u_n)]^\gamma = m(u_{n-1}, u_n),$$

so (2.3) together with the non-decreasing character of ψ lead to:

$$m(u_{n+1}, u_n) \leq \psi([m(u_{n-1}, u_n)]^{1-\gamma} . [m(u_n, u_{n+1})]^\gamma) \leq \psi\left(m(u_{n-1}, u_n)\right)$$

By repeating this argument, we get:

$$m(u_{n+1}, u_n) \leq \psi\left(m(u_{n-1}, u_n)\right) \leq \psi^2\left(m(u_{n-2}, u_{n-1})\right) \leq \cdots \leq \psi^n\left(m(u^0, u^1)\right) \quad (2.7)$$

Taking $n \to \infty$ in (2.7) and using the fact $lim_{n\to\infty}\psi^n(t) = 0$ for each $t > 0$, we conclude that $\ell = 0$, that is,

$$lim_{n\to\infty} m(u_{n+1}, u_n) = 0$$

Now, we prove that the sequence $\{u_n\}$ for $n \in N$ is an m −cauchy sequence. Take $m, n \in N$ with $m > n > n_0$. First notice that the following fact above triangular inequality of m −metric space.

$$m(u,v) - m_{u,v} \le \left(m(u,w) - m_{u,w}\right) + \left(m(w,v) - m_{w,v}\right)$$
$$\le m(u,w) + m(w,v), for\ all\ u,v,w \in X$$

Thus it is clear that

$$m(u_n, u_m) - m_{u_n,u_m} \le m(u_n, u_{n+1}) + m(u_{n+1}, u_{n+2}) + \cdots + m(u_m, u_{m+1})$$
$$< \sum_{i=n}^{\infty} m(u_i, u_{i+1})$$
$$\le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}}$ is converges, it implies that $m(u_n, u_m) - m_{u_n,u_m}$ converges as $m, n \to \infty$. Now, if $M_{u_n,u_m} = 0$, then $m_{u_n,u_m} = 0$ which implies that $M_{u_n,u_m} - m_{u_n,u_m} = 0$, so we may assume that $M_{u_n,u_m} > 0$. Then

$$m(u_n, u_n) \le \psi\left(m(u_{n-1}, u_{n-1})\right) \le \psi^2\left(m(u_{n-1}, u_{n-1})\right) \le \cdots \le \psi^{\mathrm{n}}\left(m(u^0, u^0)\right) \quad (2.8)$$

Taking $n \to \infty$ in (2.8) and using that fact $lim_{n\to\infty}\psi^{\mathrm{n}}(t) = 0$ for $t > 0$. We deduce that $lim_{n\to\infty}m(u_n, u_n) = 0$. Thus there exist $n_1 \in N$ such that $m(u_n, u_n) \le 1$ for $n > n_1$. Consequently, we have $m(u_n, u_n) < \frac{1}{i^{\frac{1}{h}}}$ for all $n > n_1$.

Therefore, we obtain

$$m(u_n, u_n) - m(u_m, u_m) \le m(u_n, u_n) + m(u_{n+1}, u_{n+1}) + \cdots + m(u_m, u_m),$$
$$< \sum_{i=n}^{\infty} m(u_i, u_i)$$
$$\le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}}$ is convergent, we conclude that $m(u_n, u_n) - m(u_m, u_m)$ converges as $m, n \to \infty$, which implies that $M_{u_n,u_m} - m_{u_n,u_m}$ converges as desired. Therefore $\{u_n\}$ is an m −cauchy sequence in X. Since (X, m) is an complete m-metric space, $\{u_n\}$ converges to some $z^* \in X$.

$$lim_{n\to\infty}(u_n, Tu_n) \to 0$$

Since $m(u_n, u_{n+1}) < m(z^*, z^*)$. Now using the fact that $m_{u_n,Tu_n} \to 0$ by lemma (1) and (2) we conclude that the

$$m(z^*, Tz^*) = m_{z^*,Tz^*} = m(Tz^*, Tz^*) \quad (2.9)$$

That is $Tz^* = z^*$

In this theorem also use the m-metric space instead of metric space theorem 2 in [2].

In what follows, we replace the continuity criteria by a weakened condition (H).

Theorem 2: Suppose a self mapping $T: X \to X$ is ϖ −orbital admissible and forms an ϖ −interpolative Ciric-Reich-Rus-type contraction on a complete m-metric space (X, m). Suppose also that the condition (H) is fulfilled. If there exist $u_0 \in X$ such that $\varpi(u_0, Tu_0) \ge 1$, then T possesses a fixed point in X.

Proof By the proof By the proof of Theorem (1) verbatim, we conclude that the constructed sequence $\{u_n\}$ is Cauchy and (1.9) holds. Suppose the condition (H) holds. We argue by contradiction by assuming that $u \ne Tu$. Recall that $u_{n(k)} \ne Tu_{n(k)}$ for each $k \ge 0$. Due to (H), there exist a partial subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $\varpi\left(u_{n(k)}, Tu\right) \ge 1$ for all k. Since $\left\{m\left(u_{n(k)}, u\right)\right\} \to 0, \left\{m\left(u_{n(k)}, Tu_{n(k)}\right)\right\} \to 0$ and $m(u, Tu) > 0$, there is $N \in \mathbb{N}$ such that, for each $k \ge N$,

$$m\left(u_{n(k)}, u\right) \le m(u, Tu)\ and\ m\left(u_{n(k)}, Tu_{n(k)}\right) \le m(u, Tu)$$

Taking $u = u_{n(k)}$ and $v = u$ in (2.1), we get that:

$$m\left(u_{n(k)+1}, Tu\right) \le \varpi\left(u_{n(k)}, u\right) m\left(Tu_{n(k)}, Tu\right)$$
$$\le \psi\left(\left[m\left(u_{n(k)}, u\right)\right]^{\beta} . \left[m\left(u_{n(k)}, Tu_{n(k)}\right)\right]^{\gamma} . [m(u, Tu)]^{1-\gamma-\beta}\right) \quad (2.10)$$

As ψ is non-decreasing, it follows from (2.10) that:

$$m(u_{n(k)+1}, Tu) \le \psi\left([m(u, Tu)]^{\beta} . [m(u, Tu)]^{\gamma} . [m(u, Tu)]^{1-\gamma-\beta}\right) = \psi(m(u, Tu)).$$

Letting $k \to \infty$, we find that:

$$0 < m(u, Tu) \le \psi(m(u, Tu)) < m(u, Tu),$$

which is a contradiction. Thus, $u = Tu$.

In what follows, we introduce tha notion of ϖ −interpolative Kannan-type contractions.

Definition The self-mapping T on the m −metric space (X, m) is called an ϖ −interpolative Kannan-type contractions if there exist $\psi \in \Psi$, $\varpi: X \times X \to [0, \infty)$ and $\beta \in (0,1)$ such that:

$$\varpi(u, v) m(Tu, Tv) \le \psi\left([m(u, v)]^{\beta} . [m(v, Tv)]^{1-\beta}\right)$$

for all $u, v \in X \backslash Fix(T)$.

The following one is our second main result.

Theorem 3: Let a self-mapping $T: X \to X$ is ϖ −orbital admissible and forms an ϖ −interpolative Kannan-type contraction on a complete m −metric space (X, m). Assume also that either T is continuous on (X,m) or (H) holds. If there exist $u_0 \in X$ such that $\varpi(u_0,Tu_0) \geq 1$, then T possesses a fixed point in X.

We skipped the proof due to the verbatim proof of Theorem 1.

By considering $\varpi(u,v) = 1$, in Theorem 1, we state the following.

Corollary 1: Let T is self-mapping on a complete m −metric space (X,m) such that:

$$m(Tu,Tv) \leq \psi\left([m(u,v)]^{\beta}.[m(u,Tu)]^{\gamma}.[m(v,Tv)]^{1-\gamma-\beta}\right)$$

for all $u,v \in X \backslash Fix(T)$, where $\gamma,\beta > 0$ are positive reals satisfying $\gamma+\beta < 1$. Then, T admit a fixed point.

Corollary 2: Let T is self-mapping on a complete m-metric space (X,m) such that:

$$m(Tu,Tv) \leq \psi\left([m(u,v)]^{\beta}.[m(v,Tv)]^{1-\beta}\right),$$

for all $u,v \in X \backslash Fix(T)$, where $0 < \beta < 1$. Then, T admit a fixed point.

Taking $\psi(t) = \lambda t$ (where $\lambda \in [0,1]$) in Corollary 1, we state:

Corollary 3: Let T is self-mapping on a complete m-metric space (X,m) such that:

$$m(Tu,Tv) \leq \lambda.[m(u,v)]^{\beta}.[m(u,Tu)]^{\gamma}.[m(v,Tv)]^{1-\gamma-\beta},$$

for all $u,v \in X \backslash Fix(T)$, where γ,β are positive reals satisfying $\gamma+\beta < 1$ and $\lambda \in [0,1)$. Then, T admit a fixed point.

Taking $\psi(t) = \lambda t$ (where $\lambda \in [0,1]$) in Corollary 2, we state:

Corollary 4: Let T is self-mapping on a complete m −metric space (X,m) such that:

$$m(Tu,Tv) \leq \lambda.[m(u,v)]^{\beta}.[m(v,Tv)]^{1-\beta},$$

or all $u,v \in X \backslash Fix(T)$, where $0 < \beta < 1$ and $\lambda \in [0,1)$. Then, T admit a fixed point.

Remark 1: Corollary 3 corresponds to Corollary 2.1 in [2].

Let $(X,m,\preccurlyeq)$ be a complete partially-ordered m −metric space. Let us consider the following condition.

(G) If $\{u_n\}$ is a sequence in X such that $u_n \preccurlyeq u_{n+1}$ for each n and $u_n \to u \in X$ as $n \to \infty$, then there exists $\{u_{n(k)}\}$ from $\{u_n\}$ such that $u_{n(k)} \preccurlyeq u$ for each k.

Following [1], we may state the following consequences of Theorem 1.

Corollary 5: Let $(X,m,\preccurlyeq)$ be a complete partially-ordered m −metric space. Let $T: X \to X$ be the mapping such that:

$$\varpi(u,v)m(Tu,Tv) \leq \psi\left([m(u,v)]^{\beta}.[m(u,Tu)]^{\gamma}.[m(v,Tv)]^{1-\gamma-\beta}\right),$$

for all $u,v \in X \backslash Fix(T)$ with $u \preccurlyeq v$, where $\psi \in \Psi$ and $\gamma,\beta > 0$ are positive reals satisfying $\gamma+\beta < 1$. Assume that:

(i) T is non-decreasing with respect to $\preccurlyeq$;

(ii) there exist $u_0 \in X$ such that $u_0 \preccurlyeq Tu_0$;

(iii) either T is continuous on (X,m) or (G) holds.

Then, T has a fixed point in X.

Proof It suffices to take, in Theorem 1,

$$\varpi(u,v) = \begin{cases} 1 & if \quad (u \preccurlyeq v)\ or\ (v \preccurlyeq u) \\ 0 & otherwise \end{cases}$$

Corollary 6: Let $(X,m,\preccurlyeq)$ be a complete partially-ordered m −metric space. Let $T: X \to X$ be tha mapping such that:

$$\varpi(u,v)m(Tu,Tv) \leq \psi\left([m(u,v)]^{\beta}.[m(v,Tv)]^{1-\beta}\right),$$

for all $u,v \in X \backslash Fix(T)$ with $u \preccurlyeq v$, where $\psi \in \Psi$ and $0 < \beta < 1$. Assume that:

(i) T is non-decreasing with respect to $\preccurlyeq$;

(ii) there exist $u_0 \in X$ such that $u_0 \preccurlyeq Tu_0$;

(iii) either T is continuous on (X,m) or (G) holds.

Then, T has a fixed point in X.

Proof We take in Theorem 3,

$$\varpi(u,v) = \begin{cases} 1 & if \quad (u \preccurlyeq v)\ or\ (v \preccurlyeq u) \\ 0 & otherwise \end{cases}$$

Corollary 7: Suppose that the subsets A_1 and A_2 of a complete m −metric space (X,m) are closed. Suppose also that $T: A_1 \cup A_2 \to A_1 \cup A_2$ satisfies:

$$\varpi(u,v)m(Tu,Tv) \leq \psi\left([m(u,v)]^{\beta}.[m(u,Tu)]^{\gamma}.[m(v,Tv)]^{1-\gamma-\beta}\right),$$

for all $u \in A_1$ and $v \in A_2$ such that $u,v \notin Fix(T)$, where $\psi \in \Psi$ and $\gamma,\beta > 0$ are positive reals satisfying $\gamma+\beta < 1$. If $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$, then there exist a fixed point of T in $A_1 \cap A_2$.

Proof It suffices to take, in Theorem 1,

$$\varpi(u,v) = \begin{cases} 1 & if \quad (A_1 \times A_2) \cup (A_1 \times A_2) \\ 0 & otherwise \end{cases}$$

Corollary 8: Suppose that the subsets A_1 and A_2 of a complete m −metric space (X, m) are closed. Suppose also that $T: A_1 \cup A_2 \to A_1 \cup A_2$ satisfies:

$$\varpi(u,v)m(Tu,Tv) \le \psi\big([m(u,v)]^{\beta}.[m(v,Tv)]^{1-\beta}\big),$$

for all $u \in A_1$ and $v \in A_2$ such that $u, v \notin Fix(T)$, where $\psi \in \Psi$ and $\gamma, \beta > 0$ are positive reals satisfying $\gamma + \beta < 1$. If $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$, then there exist a fixed point of T in $A_1 \cap A_2$.

Proof It suffices to take, in Theorem 3,

$$\varpi(u,v) = \begin{cases} 1 & if \quad (A_1 \times A_2) \cup (A_1 \times A_2) \\ & 0 \quad otherwise \end{cases}$$

Exapmle 1: Let us consider the set $X = [0,1]$ endowed with $m(u,v) = \left(\frac{u+v}{2}\right)$. Let T be a self-mapping on X defined by:

$$T(u) = \left(\frac{1+u}{2}\right) \; for \; all \; u \in X.$$

Take

$$\varpi(u,v) = 1 \; for \; all \; u, v \in X.$$

Let $u, v \in X$ be such that $u \neq Tu, v \neq Tv$ and $\varpi(u,v) \ge 1$. We first show that T is ϖ −orbital admissible, if for all $u \in X$, we have $\varpi(u,Tu) \ge 1$ then this implies that $\varpi(Tu, T^2u) \ge 1$, which satisfes.

Hence, T is ϖ −orbital admissible.

Clearly, $m(u,v) = \left(\frac{u+v}{2}\right)$ is m −metric space.

Hence by Equation (2.1),

$$\varpi(u,v)m(Tu,Tv) \le \psi\big([m(u,v)]^{\beta}.[m(u,Tu)]^{\gamma}.[m(v,Tv)]^{1-\gamma-\beta}\big).$$

The above inequality hold for all $u, v \in X$ with $\beta = \left(\frac{1}{5}\right)$ and $\gamma = \left(\frac{1}{2}\right)$. We defined $\psi(u) = e^u$ for all $u \in X$.

Hence, all condition of Theorem 1 is Hold so T has a fixed point which is $T(1) = 1$.

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