International Journal of Advanced Engineering Research and Science (IJAERS)
Peer-Reviewed Journal
ISSN: 2349-6495(P) | 2456-1908(O)
Vol-11, Issue-7; Jul, 2024
Journal Home Page Available: https://ijaers.com/
Article DOI: https://dx.doi.org/10.22161/ijaers.117.6

Enhanced Integration of Kinematic free surface boundary condition to third order Accuracy for Wave Transformation Modeling

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Received: 19 May 2024,

Receive in revised form: 27 Jun 2024,

Accepted: 06 Jul 2024,

Available online: 14 Jul 2024

***Keywords*— wave amplitude function, critical wave steepness and breaker length index.**

***Abstract*— The relationship between wave amplitude and other wave constants is derived through the integration of the Kinematic Free Surface Boundary Condition (KFSBC) over time, yielding the wave amplitude function. This function presents the characteristics of wave breaking to determine the breaker length index—the ratio of breaking wave height to breaking wave length. Previous research achieved integration at a zero-order accuracy level, assuming small wave amplitude and long wave, resulting in a notably large critical breaker length index and wave steepness. In this research, KFSBC integration was advanced to second and third-order accuracies, yielding a wave amplitude function that significantly reduces the critical breaker length index and wave steepness compared to zero-order integration. Subsequently, utilizing the third-order wave amplitude function, a comprehensive wave transformation model incorporating shoaling, breaking, refraction, and diffraction was developed, demonstrating robust model performance.*

I. INTRODUCTION

This research aimed to improve the accuracy of critical wave steepness and breaking wave steepness estimation. The breaking wave steepness equation is derived from the wave amplitude function obtained through the integration of KFSBC over time (Hutahaean, 2024). Critical wave steepness is determined by identifying the maximum wave amplitude in the dispersion equation, where the dispersion equation results from applying the wave amplitude function to the Euler conservation momentum equation. Therefore, both critical wave steepness and breaking wave steepness are derived from the wave amplitude function.

Hutahaean (2024) conducted integration with zeroth-order accuracy, yielding a wave amplitude function that produced significant critical and breaking wave steepness. This research extended the integration to second and third-order accuracy levels, examining resulting critical wave steepness and breaking wave steepness.

Additionally, Hutahaean (2024) developed a water wave transformation model based on the wave amplitude function. This research further refined the model with a new wave amplitude function, investigating breaking parameters such as breaking wave height and depth. The findings demonstrate that higher integration accuracy improves the wave amplitude function's effectiveness in predicting these parameters, validating the chosen integration method's efficacy.

II. COMPLETE SOLUTION OF THE LAPLACE EQUATION

The complete solution of the weighted Laplace Equation using the separation variable method, Hutahaean (2024) in deep water where the bottom slope has no effect, is

$\phi(x,z,t) =$

$G\ (\cos k_x x + \sin k_x x) \cosh k_z(h+z) \sin \sigma t$ .....(1)

$\phi(x,z,t)$: velocity potential

G : wave constant

k_x : wave number on the horizontal-x axis

$$k_x = \frac{k}{\sqrt{\gamma_x}}$$

k_z : wave number on the vertical-z

$$k_z = \frac{k}{\sqrt{\gamma_z}}$$

k is the general wave number

γ_x and γ_z is weighting coefficient in weighted Taylor series, see section (3).

$\sigma = \frac{2\pi}{T}$ is angular frequency, where T is wave period.

h : water depth

Equation (1) is carried out at the characteristic point where $\cos k_x x = \sin k_x x$,

$\phi(x,z,t) = 2\ G\ \cos k_x x \cosh k_z(h+z) \sin \sigma t$.(2)

III. WEIGHTED KINEMATIC FREE SURFACE BOUNDARY CONDITION

a. Weighted Taylor series

The Weighted Taylor series refers to a modified Taylor series truncated to first-order terms, where each first-order term incorporates a specific weighting coefficient in place of higher-order terms (Hutahaean, 2023).

Weighted Taylor series for function $f = f(x,t)$,

$$f(x+\delta x, t+\delta t) = f(x,t) + \gamma_{t,2}\delta t \frac{\partial f}{\partial t} + \gamma_x \delta x \frac{\partial f}{\partial x} \quad ......(3)$$

Weighted Taylor series for function $f = f(x,z,t)$,

$$f(x+\delta x, z+\delta z, t+\delta t) = f(x,z,t) + \gamma_{t,3}\delta t \frac{\partial f}{\partial t} + \gamma_x \delta x \frac{\partial f}{\partial x} + \gamma_z \delta z \frac{\partial f}{\partial z} \quad ......(4)$$

$\gamma_{t,3}$, $\gamma_{t,2}$, γ_x and γ_z are weighting coefficients. The basic values of the weighting coefficient are, $\gamma_{t,3} = 3.0$, $\gamma_{t,2} = 2.0$, $\gamma_x = 1.0$ and $\gamma_z = 1$. The corrected weighting coefficient values as a function of the optimization coefficient ε are presented in Table (1) as follows.

Table (1). Corrected weighting coefficients values.

ε	$\gamma_{t,2}$	$\gamma_{t,3}$	γ_x	γ_z
0.010	1.9998	3.00465	0.99879	1.01093
0.011	1.99975	3.00563	0.99854	1.01325
0.012	1.99971	3.00671	0.99826	1.0158
0.013	1.99966	3.00788	0.99795	1.01858
0.014	1.99960	3.00915	0.99763	1.02159
0.015	1.99954	3.01052	0.99727	1.02484
0.016	1.99948	3.01198	0.9969	1.02832
0.017	1.99941	3.01355	0.99649	1.03205
0.018	1.99934	3.01521	0.99607	1.03601
0.019	1.99926	3.01697	0.99561	1.04022
0.020	1.99918	3.01883	0.99514	1.04468

b. Weighted Kinematic Free Surface Boundary Condition (KFSBC).

For the water surface elevation equation $\eta = \eta(x,t)$, the weighting Taylor series is,

$$\eta(x+\delta x, t+\delta t) = \eta(x,t) + \gamma_{t,2}\delta t \frac{\partial \eta}{\partial t} + \gamma_x \delta x \frac{\partial \eta}{\partial x}$$

The first term on the right side is moved to the left, and the Equation is divided by δt,

$$\frac{\eta(x+\delta x, t+\delta t) - \eta(x,t)}{\delta t} = \gamma_{t,2}\frac{\partial \eta}{\partial t} + \gamma_x \frac{\delta x}{\delta t}\frac{\partial \eta}{\partial x}$$

As δt approaches zero, the left side of the equation represents the speed of change in water surface elevation, which corresponds to the total vertical velocity of surface water particles.

$$w_\eta = \gamma_{t,2}\frac{\partial \eta}{\partial t} + \gamma_x u_\eta \frac{\partial \eta}{\partial x} \quad ....(5)$$

This equation is denoted as KFSBC. Here, w_η represents the vertical velocity of surface water particles, and u_η denotes the horizontal velocity at the water surface.

Equation (5) can thus be expressed as the equation describing changes in water level elevation, namely,

$$\gamma_{t,2}\frac{\partial \eta}{\partial t} = w_\eta - \gamma_x u_\eta \frac{\partial \eta}{\partial x} \quad ....(6)$$

IV. KFSBC INTEGRATION WITH 0TH ORDER ACCURACY.

Hutahaean (2024) utilized zeroth-order integration to establish the relationship between wave amplitude and wave constants G, such as wave number k and angular frequenc σ, termed as the wave amplitude function. The subsequent section will reiterate the steps involved in the zeroth-order integration process to outline the formulation of the equation for the wave amplitude function.

1.1. The formulation of wave amplitude function.

Using equation (2), the vertical water particle velocity is:

$$w=-\frac{\partial \phi}{\partial z}=-2\,Gk_z\,\cos k_x x\sinh k_z(h+z)\sin\sigma t$$

Surface vertical water particle velocity is

$$w_\eta=-2\,Gk_z\,\cos k_x x\sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t \qquad \ldots\ldots.(7)$$

Meanwhile, the horizontal water particle velocity is

$$u=-\frac{\partial \phi}{\partial x}=2Gk_x\,\sin k_x x\cosh k_z(h+z)\sin\sigma t$$

Surface horizontal water particle velocity is

$$u_\eta=2Gk_x\,\sin k_x x\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t \qquad \ldots\ldots(8)$$

The substitution of (7) and (8) number (6),

$$\gamma_{t,2}\frac{\partial\eta}{\partial t}=$$
$$-2Gk_z\cos k_x\sinh k_z\left(h+\eta(x,t)\right)\sin\sigma t$$
$$-2\gamma_x Gk_x\sin k_x\cosh k_z\left(h+\eta(x,t)\right)\sin\sigma t\frac{\partial\eta}{\partial x}$$

To derive the equation for water surface elevation, integrate this equation over time t,

$$\gamma_{t,2}\eta(x,t)=$$
$$-2Gk_z\,\cos k_x x\int\sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\;dt$$
$$-2\gamma_x Gk_x\sin k_x x\int\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\,\frac{\partial\eta}{\partial x}\,dt \qquad \ldots(9)$$

The water surface elevation equation obtained depends on the integration solution's accuracy level. Theoretical accuracy ranges from order 0 to order n, determined by the presence of $\frac{\partial^n\eta}{\partial t^n}$ in the integration solution.

Order 0 integration assumes $\frac{\partial^0\eta}{\partial t^0}$ in the results, while order 1 implies $\frac{\partial^1\eta}{\partial t^1}$, order 2 involves $\frac{\partial^2\eta}{\partial t^2}$, etc. This research was limited to order 3 accuracy.

In zero-order integration, the assumption is made that in deep water and with small wave amplitudes $\left|\sinh k_z h\left(1+\frac{\eta(x,t)}{h}\right)\right|$ and $\left|\cosh k_z h\left(1+\frac{\eta(x,t)}{h}\right)\frac{\partial\eta}{\partial x}\right|$ fluctuates very little with time, allowing it to be treated as constant and excluded from integration.

$$\gamma_{t,2}\eta(x,t)=$$
$$-2Gk_z\sinh k_z\big(h+\eta(x,t)\big)\cos k_x x\int\sin\sigma t\;dt$$
$$-2\gamma_x Gk_x\cosh k_z\big(h+\eta(x,t)\big)\sin k_x x\frac{\partial\eta}{\partial x}\int\sin\sigma t\;\;dt$$

This integration can be completed by integrating the sinusoidal function alone. After integrating, the characteristic point property is worked out, namely $\cos k_x x=\sin k_x x$ and substituted with $k_x=\frac{k}{\sqrt{\gamma_x}}$ and $k_z=\frac{k}{\sqrt{\gamma_z}}$,

$$\gamma_{t,2}\eta(x,t)=\frac{2Gk}{\sigma}\;\cosh k_z\big(h+\eta(x,t)\big)$$
$$\left(\frac{\tanh k_z\big(h+\eta(x,t)\big)}{\sqrt{\gamma_z}}+\sqrt{\gamma_x}\frac{\partial\eta}{\partial x}\right)\cos k_x x\;\cos\sigma t$$

In the integrated results, the absence of the $\frac{\partial\eta}{\partial t}$ term indicates a zero-order integration.

In deep water, where $\tanh k_z\big(h+\eta(x,t)\big)\approx 1$, $k_z\big(h+\eta(x,t)\big)=\theta\pi$. θ is referred to as deep water coefficient with $\theta>1$, for instance $\theta=1.75$.

$$\eta(x,t)=\frac{2Gk}{\sigma\gamma_{t,2}}\;\cosh\theta\pi$$
$$\left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}\;\sqrt{\gamma_x}\frac{\partial\eta}{\partial x}\right)\cos k_x x\;\cos\sigma t$$

As a periodic function, hence

$$A=\frac{2Gk}{\sigma\gamma_{t,2}}\;\cosh\theta\pi\left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}+\sqrt{\gamma_x}\frac{\partial\eta}{\partial x}\right)$$

A is wave amplitude.

Water surface elevation equation is

$$\eta(x,t)=A\cos k_x x\;\cos\sigma t \qquad \ldots(10)$$

At the characteristic point of space and time,

$$\frac{\partial\eta}{\partial x}=-\frac{k_x A}{2}$$

The wave amplitude function is

$$A=\frac{2Gk}{\sigma\gamma_{t,2}}\;\cosh\theta\pi\left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}-\frac{kA}{2}\right) \qquad \ldots(11)$$

This equation is the wave amplitude function equation resulting from zero order integration.

1.2. 0th order breaking characteristics.

In (11), breaking occurs when

$$\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}-\frac{kA}{2}=0$$

$$\frac{H_b}{L_b}=\frac{2\tanh\theta\pi}{\pi\sqrt{\gamma_z}} \qquad \ldots\ldots(12)$$

Equation (12) applies universally across all wave periods, accommodating the specific wave height relevant to each period.

1.3. 0th order dispersion equation

The dispersion equation is formulated based on surface-weighted Euler momentum conservation, excluding the convective acceleration term.

$$\gamma_{t,3}\frac{\partial u_\eta}{\partial t}=-g\frac{\partial\eta}{\partial x}$$

With u_η from (8) and η and (10), the following is obtained

$$\gamma_{t,3}\frac{\partial u_\eta}{\partial t}=2\gamma_{t,3}Gk_x\sigma\,\sin k_x x\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t$$

$$-g\frac{\partial\eta}{\partial x}=gk_xA\sin k_x x\,\cos\sigma t$$

Substitute these two equations into (13),

$$2\gamma_{t,3}G\sigma\,\cosh\theta\pi=gA$$

Wave amplitude on the right side is substituted with (11),

$$2\gamma_{t,3}G\sigma\,\cosh\theta\pi=\frac{2gGk}{\sigma\gamma_{t,2}}\,\cosh\theta\pi\left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}-\frac{kA}{2}\right)$$

$$\frac{gA}{2}\,k^2-\frac{g\tanh\theta\pi}{\sqrt{\gamma_z}}k+\gamma_{t,2}\gamma_{t,3}\sigma^2=0 \qquad \ldots\ldots(13)$$

This equation is the dispersion equation resulting from zero order integration.

The steps in formulating the wave amplitude function are:

a. Completing the integration of KFSBC

b. The integration results are collected to form an Equation:$\eta(x,t)=\ (\ldots\ldots)\cos k_x x\cos\sigma t$

c. Defining wave amplitude function A =(……), obtaining $\eta(x,t)=\ A\cos k_x x\cos\sigma t$

From this Equation, we obtain the Equation of $\frac{\partial\eta}{\partial x}$, which is then substituted into the wave amplitude function equation, to obtain the final Equation of the wave amplitude function.

V. KFSBC INTEGRATION WITH 2ND ORDER ACCURACY.

The integration will be completed until there is an element $\frac{\partial^2\eta}{\partial t^2}$. Integration is carried out using the integral inversion method (Hutahaean (2010).

a. $\int\sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\;dt$

Is defined as,

$$f(t)=\sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t$$

$$\frac{df}{dt}=-\sigma\sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t+k_z\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\;\frac{\partial\eta}{\partial t}$$

This differential equation is integrated over time t,

$$\int df=-\sigma\int\sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\;dt+k_z\int\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial t}dt$$

Integration of the left side produces $f(t)$, substitution $f(t)$and the rearranged Equation is obtained

$$\int\sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\;dt=-\frac{1}{\sigma}\sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t+\frac{k_z}{\sigma}\int\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial t}dt \qquad \ldots\ldots\ldots\ldots(14)$$

In second-order integration, the fluctuations $\left|\cosh k_z h\left(1+\frac{\eta(x,t)}{h}\right)\frac{\partial\eta}{\partial t}\right|$ with respect to time are significant enough that the integration of the two terms on the right side necessitates the use of the integral inversion method.

$$\int\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial t}dt=?$$

Is defined as,

$$f(t)=\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\frac{\partial\eta}{\partial t}$$

$$\frac{df}{dt}=k_z\sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2+\sigma\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial t}+\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\frac{\partial^2\eta}{\partial t^2}$$

Is integrated into,

$$\int df=\int k_z\sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2dt+\sigma\int\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial t}\,dt+\int\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\frac{\partial^2\eta}{\partial t^2}\,dt$$

The left side is substituted with $f(t)$ and rearranged

$$\int \cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial\eta}{\partial t}\,dt =$$
$$\frac{1}{\sigma}\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\,\frac{\partial\eta}{\partial t}$$
$$-\frac{k_z}{\sigma}\int \sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\,\left(\frac{\partial\eta}{\partial t}\right)^2 dt$$
$$-\frac{1}{\sigma}\int \cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\,\frac{\partial^2\eta}{\partial t^2}\,dt$$

In this 2nd order integration, the assumption is made that fluctuations over time t from $\left|\sinh k_z h\left(1+\frac{\eta(x,t)}{h}\right)\left(\frac{\partial\eta}{\partial t}\right)^2\right|$ *and* $\left|\cosh k_z h\left(1+\frac{\eta(x,t)}{h}\right)\frac{\partial^2\eta}{\partial t^2}\right|$ is very small and can be considered constant that it can be excluded from the integral.

$$\int \cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial\eta}{\partial t}\,dt =$$
$$\frac{1}{\sigma}\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\,\frac{\partial\eta}{\partial t}$$
$$-\frac{k_z}{\sigma}\sinh k_z\big(h+\eta(x,t)\big)\left(\frac{\partial\eta}{\partial t}\right)^2\int \sin\sigma t\;dt$$
$$-\frac{1}{\sigma}\cosh k_z\big(h+\eta(x,t)\big)\frac{\partial^2\eta}{\partial t^2}\int \sin\sigma t\;dt$$

The final integration is completed by simply integrating the sinusoidal function, obtaining the followings.

$$\int \cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial\eta}{\partial t}\,dt =$$
$$\frac{1}{\sigma}\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\,\frac{\partial\eta}{\partial t}$$
$$+\frac{k_z}{\sigma^2}\sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\left(\frac{\partial\eta}{\partial t}\right)^2$$
$$+\frac{1}{\sigma^2}\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial^2\eta}{\partial t^2}$$

Or,

$$\frac{k_z}{\sigma}\int \cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial\eta}{\partial t}\,dt =$$
$$\frac{k_z}{\sigma^2}\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\,\frac{\partial\eta}{\partial t}$$
$$+\frac{k_z^2}{\sigma^3}\sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\left(\frac{\partial\eta}{\partial t}\right)^2$$
$$+\frac{k_z}{\sigma^3}\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial^2\eta}{\partial t^2}$$

Since $k_z\big(h+\eta(x,t)\big)=\theta\pi$ and calculating the integration outcomes at the specific time point, $\cos\sigma t=\sin\sigma t$

$$\frac{k_z}{\sigma}\int \cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial\eta}{\partial t}\,dt =$$
$$\frac{1}{\sigma}\cosh\theta\pi\left(\frac{k_z}{\sigma}\frac{\partial\eta}{\partial t}+\frac{k_z^2}{\sigma^2}\tanh\theta\pi\left(\frac{\partial\eta}{\partial t}\right)^2+\frac{k_z}{\sigma^2}\frac{\partial^2\eta}{\partial t^2}\right)\cos\sigma t$$

The results of this integration are substituted into (14),

$$\int \sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\;dt = -\frac{\cosh\theta\pi}{\sigma}\left(\tanh\theta\pi-\left(\frac{k_z}{\sigma}\frac{\partial\eta}{\partial t}+\frac{k_z^2}{\sigma^2}\tanh\theta\pi\left(\frac{\partial\eta}{\partial t}\right)^2+\frac{k_z}{\sigma^2}\frac{\partial^2\eta}{\partial t^2}\right)\right)\cos\sigma t \;\ldots\ldots(15)$$

Based on the integration results, there are terms $\frac{\partial^2\eta}{\partial t^2}$ that is equivalent to $\left(\frac{\partial\eta}{\partial t}\right)^2$, classifying this integration as second-order accurate.

b. $\int \cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\,\frac{\partial\eta}{\partial x}\,dt$

Is defined as,

$$f(t)=\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial\eta}{\partial x}$$
$$\frac{df}{dt}=-\sigma\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\,\frac{\partial\eta}{\partial x}$$
$$+k_z\sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}$$
$$+\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial^2\eta}{\partial t\partial x}$$

Multiplied by dt is integrated into,

$$\int \cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\,\frac{\partial\eta}{\partial x}\,dt =$$
$$-\frac{1}{\sigma}\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial\eta}{\partial x}$$
$$+\frac{k_z}{\sigma}\int \sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}\,dt$$
$$+\frac{1}{\sigma}\int \cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\,\frac{\partial^2\eta}{\partial t\partial x}\,dt$$

In this second order integration, the assumption is made that the fluctuation $\left|\sinh k_z h\left(1+\frac{\eta(x,t)}{h}\right)\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}\right|$ with respect to time t is quite small that it can be considered constant and can be excluded from the integral.

Likewise, the fluctuation $\left|\cosh k_z h\left(1+\frac{\eta(x,t)}{h}\right)\frac{\partial^2\eta}{\partial t\partial x}\right|$ with respect to time t is considered too small that it can be excluded from the integral.

$$\int \cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\frac{\partial\eta}{\partial x}\,dt = -\frac{1}{\sigma}\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial x} + \frac{k_z}{\sigma}\sinh k_z\big(h+\eta(x,t)\big)\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}\int\cos\sigma t\;dt + \frac{1}{\sigma}\cosh k_z\big(h+\eta(x,t)\big)\frac{\partial^2\eta}{\partial t\partial x}\int\cos\sigma t\;dt$$

This integration equation can be solved by integrating $\int\cos\sigma t\;dt$ directly. Once integrated and substituting $k_z(h+\eta(x,t))=\theta\pi$, the equation is evaluated at the characteristic time point where $\cos\sigma t=\sin\sigma t$, yielding:

$$\int\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\frac{\partial\eta}{\partial x}\;dt = -\frac{1}{\sigma}\cosh\theta\pi\left(\frac{\partial\eta}{\partial x}-\frac{k_z}{\sigma}\tanh\theta\pi\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}-\frac{1}{\sigma}\frac{\partial^2\eta}{\partial t\partial x}\right)\cos\sigma t \quad \ldots(16)$$

Regarding the presence of $\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}$ and $\frac{\partial^2\eta}{\partial t\partial x}$, then the result of the integration is called the second order level of accuracy.

Equation-Equation (15) and (16) are substituted to (9),

$$\eta(x,t)=\frac{Gk_z}{\sigma\gamma_{t,2}}\cosh\theta\pi\left(\tanh\theta\pi+\left(k_zA-\frac{k_z^2A^2}{4}\tanh\theta\pi\right)\right)\cos\sigma t\cos k_x x + \frac{\gamma_x Gk_x}{\sigma\gamma_{t,2}}\cosh\theta\pi\left(\frac{\partial\eta}{\partial x}-\frac{k_z}{\sigma}\tanh\theta\pi\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}-\frac{1}{\sigma}\frac{\partial^2\eta}{\partial t\partial x}\right)\cos\sigma t\cos k_x x$$

As a periodic function, therefore

$$A=\frac{Gk_z}{\sigma\gamma_{t,2}}\cosh\theta\pi\left(\tanh\theta\pi+\left(k_zA-\frac{k_z^2A^2}{4}\tanh\theta\pi\right)\right) + \frac{\gamma_x Gk_x}{\sigma\gamma_{t,2}}\cosh\theta\pi\left(\frac{\partial\eta}{\partial x}-\frac{k_z}{\sigma}\tanh\theta\pi\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}-\frac{1}{\sigma}\frac{\partial^2\eta}{\partial t\partial x}\right)$$

obtaining

$\eta(x,t)=A\cos\sigma t\cos k_x x$

Within the characteristic point of

$$\frac{\partial\eta}{\partial x}=-\frac{k_xA}{2}$$

$$\frac{\partial\eta}{\partial t}=-\frac{\sigma A}{2}$$

$$\frac{\partial^2\eta}{\partial t\partial x}=\frac{\sigma k_xA}{2}$$

Is the differential substitution of $\eta(x,t)$ and $k_x=\frac{k}{\sqrt{\gamma_x}}$ and $k_z=\frac{k}{\sqrt{\gamma_z}}$

$$A=\frac{2Gk}{\sigma\gamma_{t,2}}\cosh\theta\pi\left(\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}+\left(\frac{1}{\gamma_z}-1\right)kA-\left(\frac{1}{\gamma_z\sqrt{\gamma_z}}+\frac{\sqrt{\gamma_x}}{\sqrt{\gamma_z}}\right)\frac{\tanh\theta\pi}{4}k^2A^2\right) \quad \ldots\ldots(17)$$

This equation is the wave amplitude function resulting from integration with 2nd order accuracy.

5.2. Breaking 2nd order characteristics

Breaking occurs when

$$\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}+\left(\frac{1}{\gamma_z}-1\right)kA-\left(\frac{1}{\gamma_z\sqrt{\gamma_z}}+\frac{\sqrt{\gamma_x}}{\sqrt{\gamma_z}}\right)\frac{\tanh\theta\pi}{4}k^2A^2=0$$

Is substituted to $k=\frac{2\pi}{L}$ and sinusoidal wave $A=\frac{H}{2}$,

$$\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}+\left(\frac{1}{\gamma_z}-1\right)\pi\left(\frac{H}{L}\right)-\left(\frac{1}{\gamma_z\sqrt{\gamma_z}}+\frac{\sqrt{\gamma_x}}{\sqrt{\gamma_z}}\right)\frac{\tanh\theta\pi}{4}\pi^2\left(\frac{H}{L}\right)^2=0 \quad \ldots(18)$$

Using equation (18), $\frac{H}{L}$ known as the breaker length index $\frac{H_b}{L_b}$ can be calculated. This equation does not include parameters for wave period or wave amplitude. Therefore, $\frac{H_b}{L_b}$ obtained is valid for all wave periods, provided the appropriate wave amplitude is considered.

5.3. 2nd order Dispersion Equation.

In the same way as the formulation of the dispersion equation in (4.3) where in this case the wave amplitude function (17) is used, the dispersion equation obtained is

$$\left(\frac{1}{\gamma_z\sqrt{\gamma_z}}+\frac{\sqrt{\gamma_x}}{\sqrt{\gamma_z}}\right)\frac{\tanh\theta\pi}{4}gA^2k^3-\left(\frac{1}{\gamma_z}-1\right)gAk^2 -\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}gk+\gamma_{t,2}\gamma_{t,3}\sigma^2=0 \quad \dots(19)$$

The equation is a third-degree polynomial that can be solved using the Newton-Raphson iteration method. Initially, terms containing k^3 are disregarded to simplify the equation to a second-degree polynomial. Once k is found from this simplified equation, it is used in the complete set of equations to calculate k accurately.

VI. KFSBC INTEGRATION WITH 3RD ORDER ACCURACY

In this section the integration process is not discussed in its entirety, just an example. In second order integration there are integration results,

$$\int \cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial t}\,dt = \frac{1}{\sigma}\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\frac{\partial\eta}{\partial t} -\frac{k_z}{\sigma}\int \sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2 dt -\frac{1}{\sigma}\int \cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\frac{\partial^2\eta}{\partial t^2}\,dt$$

In second-order integration, the two integrations on the right side are solved by integrating only the sinusoidal elements. However, in third-order integration, the two right-hand side integrations are solved using the integral inversion method. For example,

$$\frac{k_z}{\sigma}\int \sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2 dt$$

$$f(t)=\sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2$$

$$\frac{df}{dt}=k_z\cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\left(\frac{\partial\eta}{\partial t}\right)^3 -\sigma\sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2 +2\sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial t}\frac{\partial^2\eta}{\partial t^2}$$

This equation is multiplied with dt and is integrated into,

$$\int df = k_z\int \cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\left(\frac{\partial\eta}{\partial t}\right)^3 dt -\sigma\int \sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2 dt +2\int \sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial t}\frac{\partial^2\eta}{\partial t^2}\,dt$$

Substitute $f(t)$ on the left side and is moved to the right, while the second term on the right side is moved to the left,

$$\int \sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2 dt = -\frac{1}{\sigma}\sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2 +\frac{k_z}{\sigma}\int \cosh k_z\big(h+\eta(x,t)\big)\cos\sigma t\left(\frac{\partial\eta}{\partial t}\right)^3 dt +\frac{2}{\sigma}\int \sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial t}\frac{\partial^2\eta}{\partial t^2}\,dt$$

The assumption is that fluctuations with time from: $\left|\cosh k_zh\left(1+\frac{\eta(x,t)}{h}\right)\left(\frac{\partial\eta}{\partial t}\right)^3\right|$ and $\left|\sinh k_zh\left(1+\frac{\eta(x,t)}{h}\right)\frac{\partial\eta}{\partial t}\frac{\partial^2\eta}{\partial t^2}\right|$ is very small and can be considered constant, that it can be excluded from integration, and what is integrated into, is only the sinusoidal element. Obtaining,

$$\frac{k_z}{\sigma}\int \sinh k_z\big(h+\eta(x,t)\big)\sin\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2 dt = -\frac{k_z}{\sigma^2}\sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\left(\frac{\partial\eta}{\partial t}\right)^2 +\frac{k_z^2}{\sigma^3}\cosh k_z\big(h+\eta(x,t)\big)\sin\sigma t\left(\frac{\partial\eta}{\partial t}\right)^3 +\frac{2k_z}{\sigma^3}\sinh k_z\big(h+\eta(x,t)\big)\cos\sigma t\frac{\partial\eta}{\partial t}\frac{\partial^2\eta}{\partial t^2}$$

After all the integrations are completed then add them up and carry out the process as in 2nd order integration, we get,

$$\eta(x,t)=\frac{2Gk\cosh\theta\pi}{\sigma\gamma_{t,2}}\alpha(k,A)\cos k_xx\cos\sigma t$$

$$\alpha(k,A)=\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}+\left(\frac{3}{2\gamma_z}-\frac{3}{2}\right)kA -\frac{k^2A^2}{2\sqrt{\gamma_x}\sqrt{\gamma_z}}\tanh\theta\pi$$

$$+\left(\frac{1}{\gamma_z^2}-\frac{1}{\gamma_z}\right)\frac{k^3A^3}{8}$$

Where the wave amplitude function is

$$A=\frac{2Gk\cosh\theta\pi}{\sigma\gamma_{t,2}}\alpha(k,A) \quad .....(20)$$

Breaking occurs when,

$$\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}+\left(\frac{3}{2\gamma_z}-\frac{3}{2}\right)kA-\frac{(kA)^2}{2\sqrt{\gamma_x}\sqrt{\gamma_z}}\tanh\theta\pi$$
$$+\left(\frac{1}{\gamma_z^2}-\frac{1}{\gamma_z}\right)\frac{(kA)^3}{8}=0 \quad ......(21)$$

From this equation, the kA value can be calculated. It is a third-degree polynomial equation, which can be solved using the Newton-Raphson iteration method. Initially, the kA value is estimated by disregarding the terms. After obtaining an approximate kA value using this simplified equation, it can then be refined using the complete set of equations. For example,

$kA=\lambda$

Considering $k=\frac{2\pi}{L}$ and $H=2A$,

$$\frac{H_b}{L_b}=\frac{\lambda}{\pi}$$

Dispersion equation with wave amplitude function obtained from 3rd order integration is,

$$\left(\frac{1}{\gamma_z^2}-\frac{1}{\gamma_z}\right)\frac{A^3}{8}k^4-\frac{\tanh\theta\pi\,A^2}{2\sqrt{\gamma_x}\sqrt{\gamma_z}}k^3+\left(\frac{1}{2\gamma_z}-\frac{3}{2}\right)Ak^2$$
$$+\frac{\tanh\theta\pi}{\sqrt{\gamma_z}}k-\frac{\gamma_{t,3}\gamma_{t,3}\,\sigma^2}{g}=0 \quad ....(22)$$

This equation is a 4th-degree polynomial, which can be solved using the Newton-Raphson iteration method. The initial iteration value is obtained by disregarding the terms with powers of 4 and 3.

VII. WAVELENGTH, CRITICAL WAVE STEEPNESS AND BREAKER LENGTH INDEX

In this section, a comparative research of wavelength, maximum wave height, and critical wave steepness is conducted. The research utilizes the dispersion equation with an optimization coefficient ε of 0.01 for the weighting coefficient.

In the dispersion equation derived from zero-order integration using the wave amplitude function, the maximum wave height corresponds to the wave height at which the determinant value of equation (13) becomes zero.

For the dispersion equation (19) obtained from second-order integration using the wave amplitude function, using an excessively large wave height results in a negative wave number. Therefore, the maximum wave height in equation (19) is the largest wave height that does not produce a negative wave number.

Similarly, in the dispersion equation (22) resulting from third-order integration with the wave amplitude function, using a wave height that is too large also yields a negative wave number. Thus, the maximum wave height in equation (22) is the highest wave height before encountering a negative wave number.

There are two types of wave numbers: the horizontal wave number k_x and vertical wave number k_z, where

$k_x=\frac{k}{\sqrt{\gamma_x}}$ and $k_z=\frac{k}{\sqrt{\gamma_z}}$

Therefore, two wavelengths are,

Horizontal wavelength : $L_x=\frac{2\pi}{k_x}=\frac{2\pi}{k}\sqrt{\gamma_x}=L\sqrt{\gamma_x}$

Vertical wavelength : $L_z=\frac{2\pi}{k_z}=\frac{2\pi}{k}\sqrt{\gamma_z}=L\sqrt{\gamma_z}$

Table (2) presents the maximum wave height of (13)

Table (2). Maximum wave height from (13)

T (sec)	H_{max} (m)	L_x (m)	L_z (m)	$\frac{H_{max}}{L_x}$
6	1.453	4.64	4.699	0.313
7	1.977	6.401	6.481	0.309
8	2.583	8.272	8.377	0.312
9	3.269	10.487	10.619	0.312
10	4.036	12.914	13.077	0.313
11	4.883	15.708	15.906	0.311
12	5.812	18.56	18.795	0.313
13	6.821	21.79	22.064	0.313
14	7.911	25.188	25.506	0.314
15	9.081	29.056	29.422	0.313

Table (3). Maximum wave height from (19)

T (sec)	H_{max} (m)	L_x (m)	L_z (m)	$\frac{H_{max}}{L_x}$
6	1.565	6.169	6.246	0.253
7	2.13	8.41	8.516	0.253
8	2.782	10.988	11.126	0.253
9	3.522	13.76	13.934	0.256
10	4.348	17.027	17.241	0.255

11	5.261	20.618	20.878	0.255
12	6.261	24.542	24.852	0.255
13	7.348	28.799	29.162	0.255
14	8.522	33.389	33.81	0.255
15	9.783	38.31	38.793	0.255

Based on the data presented in Tables 2, 3, and 4, it is evident that the difference between the horizontal wavelength L_x and the vertical wavelength $L_z$is minimal.

Table (4). Wave height maximum from (22)

T (sec)	H_{max} (m)	L_x (m)	L_z (m)	$\frac{H_{max}}{L_x}$
6	1.53	6.201	6.279	0.247
7	2.082	8.468	8.575	0.246
8	2.72	11.023	11.162	0.247
9	3.444	13.839	14.013	0.249
10	4.252	17.07	17.285	0.249
11	5.146	20.474	20.733	0.251
12	6.124	24.437	24.745	0.251
13	7.186	28.835	29.199	0.249
14	8.336	33.015	33.432	0.252
15	9.568	38.292	38.775	0.25

As the integration order increases, the critical wave steepness decreases. Specifically, the difference between the critical wave steepness values derived from equations (19) and (22) is relatively small, typically in the third decimal place. This suggests that increasing the integration order to 4th order would yield a critical wave steepness not significantly different from that obtained at 3rd order. In general, the critical wave steepness is estimated to be 0.250.

Toffoli et al. (2010) propose a critical wave steepness of 0.170, while suggesting it could reach up to 0.20. It is noted that the wave steepness calculated at 3rd order accuracy closely approximates the criteria proposed by Toffoli et al.

The comparison of the breaker wave steepness $\frac{H_b}{L_b}$from equations (12), (18), and (21), using parameters $\theta = 1.75$ and $\varepsilon = 0.01$, progresses from the lowest to highest integration order.

Dari (12), $\frac{H_b}{L_b} = 0.629$

Dari (18), $\frac{H_b}{L_b} = 0.446$

Dari (21), $\frac{H_b}{L_b} = 0.437$

The critical wave steepness in deep water differs from the wave steepness at the breaking point, $\frac{H_b}{L_b}$, where breaking waves concentrate a significant amount of energy. Among the three $\frac{H_b}{L_b}$ values, the smallest is obtained from equation (21), which corresponds to 3rd order integration. The difference between $\frac{H_b}{L_b}$from equation (18) and $\frac{H_b}{L_b}$from equation (21) is negligible, indicating that increasing the integration accuracy will not lead to substantial changes.

In conclusion, the breaking wave steepness $\frac{H_b}{L_b}$is generally estimated to be 0.437

VIII. WAVE TRANSFORMATION MODEL

The wave amplitude function equation, which establishes the relationship among various wave constants, serves as a foundational equation for developing wave transformation models (Hutahaean, 2024). By deriving a new wave amplitude function, different wave transformation phenomena can be modeled.

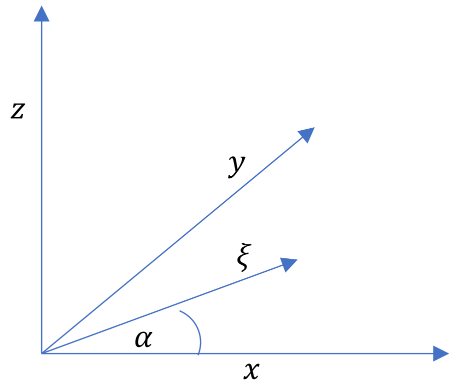

Fig (1)Axis system for wave transformation modeling.

The wave transformation model is constructed within the coordinate system depicted in Figure (1), where the wave propagates along the ξ axis in the (x, y) plane, forming an angle α with the horizontal x.

8.1 Shoaling-breaking model.

Shoaling breaking equations were developed for waves moving on the ξ-axis. The wave amplitude function resulting from 3rd order integration, namely (20), is written as,

$$A = Gk\,\lambda(k, A) \qquad \text{…..(23)}$$

$$\lambda(k, A) = \frac{2cosh\,\theta\pi}{\sigma\gamma_{t,2}}\left(\frac{tanh\,\theta\pi}{\sqrt{\gamma_z}} + \left(\frac{3}{2\gamma_z} - \frac{3}{2}\right)kA\right) + \frac{2cosh\,\theta\pi}{\sigma\gamma_{t,2}}\left(-\frac{k^2A^2}{2\sqrt{\gamma_x}\sqrt{\gamma_z}}tanh\,\theta\pi + \left(\frac{1}{\gamma_z^2} - \frac{1}{\gamma_z}\right)\frac{k^3A^3}{8}\right)$$

Equation (23) is differentiable to axis-ξ,

$\frac{dA}{d\xi} = \left(G\frac{dk}{d\xi} + k\frac{dG}{d\xi}\right)\lambda(k,A)$ …..(24)

Where considering the law of conservation of wave number which will be shown in the next section, then $\frac{d\lambda}{d\xi} = 0$.

The conservation of energy equation (Hutahaean (2024) applies,

$$G\frac{\partial k}{\partial \xi} + 2k\frac{\partial G}{\partial \xi} = 0$$

This equation is written as,

$$k\frac{\partial G}{\partial \xi} = -\frac{G}{2}\frac{\partial k}{\partial \xi}$$

Substituted to (24), obtaining

$\frac{dA}{d\xi} = \frac{G}{2}\frac{dk}{d\xi}\lambda(k,A)$ ……(25)

This equation is multiplied by $\frac{k}{k}$, Considering (23), the following is obtained

$\frac{dA}{d\xi} = \frac{A}{2k}\frac{dk}{d\xi}$ …..(26)

Wave number conservation (Hutahaean (2024)),

$$\frac{dk\left(h + \frac{A}{2}\right)}{d\xi} = 0$$

This equation is expressed as,

$$\frac{dA}{d\xi} = -\frac{2}{k}\left(h + \frac{A}{2}\right)\frac{dk}{d\xi} - 2\frac{dh}{d\xi}$$

Substituting the left side with (26)

$$\frac{A}{2k}\frac{dk}{d\xi} = -\frac{2}{k}\left(h + \frac{A}{2}\right)\frac{dk}{d\xi} - 2\frac{dh}{d\xi}$$

Therefore,

$\frac{dk}{d\xi} = -\frac{4k}{(4h+3A)}\frac{dh}{d\xi}$ ….(27)

The equation (25) is used to calculate changes in wave number k. To compute changes in wave amplitude, equation (26) should be utilized. Equation (26) incorporates breaking characteristics through $\lambda(k,A)$, where the values of k and A determine its behavior.

As water depth decreases, both k and A increase, causing λ to decrease continuously until it reaches zero, signifying the onset of breaking. Post breaking, λ turns negative, resulting in a gradual reduction of wave amplitude until it diminishes completely.

For a wave moving from point ξ to point $\xi + \delta\xi$, the change in k and A is

$$k_{\xi+\delta\xi} = k_\xi + \delta\xi\,\frac{dk}{d\xi}$$

$$A_{\xi+\delta\xi} = A_\xi + \delta\xi\,\frac{dA}{d\xi}$$

The equation for calculating changes in G is formulated using the energy conservation equation which can be written as.

$$\frac{\partial G}{G} = -\frac{1}{2}\frac{\partial k}{k}$$

Equation ini Is integrated into,,

$$\int_{\xi}^{\xi+\delta\xi}\frac{\partial G}{G} = -\frac{1}{2}\int_{\xi}^{\xi+\delta\xi}\frac{\partial k}{k}$$

$$\ln G_{\xi+\delta\xi} = \ln G_\xi - \frac{1}{2}\left(\ln k_{\xi+\delta\xi} - \ln k_\xi\right)$$

$$G_{\xi+\delta\xi} = e^{\ln G_\xi - \frac{1}{2}\left(\ln k_{\xi+\delta\xi} - \ln k_\xi\right)}$$

8.2. Outcome of the shoaling-breaking model.

The shoaling-breaking model examines waves with a period of 8 seconds and a significant wave height, $H = 2.72$ m (See Table 4), propagating over a bottom slope characterized by $\frac{dh}{dx} = -0.02$.

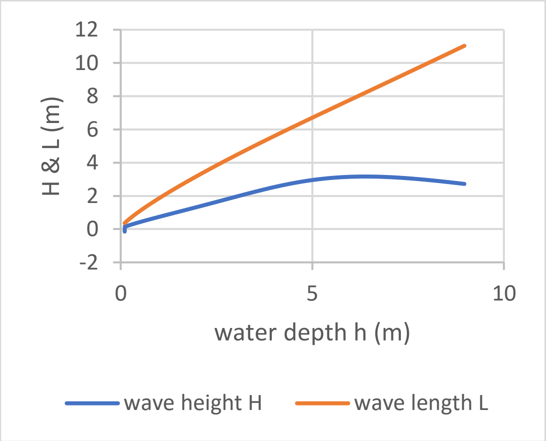

Fig (2) Shoaling-breaking analysis

The calculation employs parameters $\theta = 1.75$ and $\varepsilon = 0.01$. Fig (2) illustrates the outcomes of the shoaling and breaking analysis.

The shoaling-breaking analysis results are depicted in Figure (2), where the breaker height H_b =3.165 m, breaker depth h_b =6.387 m, and the breaker depth index $\frac{H_b}{h_b} = 0.496$. This differs notably from McCowan's (1894) criterion, where $\frac{H_b}{h_b} = 0.78$.

According to Komar and Gaughan (1972), the breaking wave height is given by:

$$H_b = 0.39\,g^{1/5}(T_0 H_0^2)^{2/5}$$

Using $T_0 = 8.0$ sec., $H_0 = 2.72$ m, $g = 9.81$ m/sec^2, $H_b = 3.15$ m. The breaking wave height from the model closely aligns with the Komar and Gaughan (1972) equation.

Next, we will examine the results of the shoaling-breaking model across several wave periods, focusing on the maximum wave height in each period as detailed in Table (4). The calculations maintain consistent parameters $\theta = 1.75$ and $\varepsilon = 0.01$. The model outcomes are summarized in Table (5).

Table (5) Shoaling breaking outcomes

T (sec)	H_0 (m)	H_b (m)	h_b (m)	$\frac{H_b}{h_b}$	H_{b-KG} (m)
6	1.53	1.781	3.593	0.496	1.772
7	2.082	2.426	4.896	0.496	2.411
8	2.72	3.165	6.387	0.496	3.15
9	3.444	3.995	8.06	0.496	3.988
10	4.252	4.93	9.95	0.496	4.923
11	5.146	5.945	11.996	0.496	5.958
12	6.124	7.081	14.289	0.496	7.091
13	7.186	8.331	16.811	0.496	8.321
14	8.334	9.662	19.499	0.496	9.65
15	9.568	11.081	22.363	0.496	11.079

Based on the results of the shoaling-breaking model, $\frac{H_b}{h_b}$ remains constant across all wave periods, specifically $\frac{H_b}{h_b} = 0.496$. The breaker height H_b closely matches the Komar-Gaughan (1972) breaker height H_{b-KG}, with a minimal difference typically in the second decimal place.

To achieve $\frac{H_b}{h_b}$ closer to McCowan's (1894) criterion, the model is executed using a deep water coefficient $\theta = 1.182$.

Table (6) presents the results of the shoaling-breaking analysis

T (sec)	H_0 (m)	H_b (m)	h_b (m)	$\frac{H_b}{h_b}$	H_{b-KG} (m)
6	1.528	1.779	2.28	0.781	1.77
7	2.08	2.42	3.101	0.78	2.409
8	2.718	3.152	4.038	0.78	3.148
9	3.44	3.988	5.111	0.78	3.984
10	4.246	4.932	6.32	0.78	4.918
11	5.138	5.965	7.645	0.78	5.951
12	6.116	7.085	9.08	0.78	7.083
13	7.178	8.313	10.652	0.78	8.313
14	8.324	9.65	12.368	0.78	9.641
15	9.556	11.073	14.192	0.78	11.068

It shows a reduction in H_0 and H_b with a relatively minor decrease, while there is a significant decrease in breaker depth h_b, resulting $\frac{H_b}{h_b} = 0.78$, aligning with McCowan's criterion.

The initial criterion for determining θ is based on wave number conservation, where $\frac{dk(h+z)}{dx} = 0$, implying $k(h+z) = constant$. This condition is crucial for solving the Laplace equation using separation of variables. For deep water, $\tanh k\left(h+\frac{A}{2}\right) \approx \tanh kh \approx 1.$, leading to $k\left(h+\frac{A}{2}\right) \approx kh = \theta\pi$, where $\tanh \theta\pi \approx 1$. Typically, $\theta \geq 1.75$ satisfies this condition. However, from the shoaling-breaking model results, $k\left(h+\frac{A}{2}\right)$ or $\approx kh$ does not need to be exactly 1. With $\theta = 1.182$, we find $\tanh \theta\pi = 0.988104$, which is slightly less than 1 but still close.

To achieve $\frac{H_b}{h_b} = 0.8$, one could consider $\theta = 1.158$. However, determining the most suitable θ and $\frac{H_b}{h_b}$ values requires additional data, including laboratory and analytical results.

By adhering to McCowan's breaker depth index (1894), the deep water depth is

$$h_0 = \frac{1.182\pi}{k_0} - \frac{A_0}{2}$$

h_0 is deep water depth , A_0 is deep water wave amplitude and k_0 is deep water wave number.

8.2 Refraction-Diffraction Model

The shoaling-breaking analysis discussed focuses on wave transformation along the direction of wave propagation, specifically along the ξ axis. Refraction-diffraction analysis, on the other hand, examines wave transformations within a plane, typically the (x, y) plane. The changes in wave number k and wave amplitude A along the horizontal axis x, as depicted in Figure 1, are related to the variables ξ and x.

$$\frac{\partial k}{\partial x} = \frac{\partial k}{\partial \xi}\cos\alpha$$

$$\frac{\partial A}{\partial x} = \frac{\partial A}{\partial \xi}\cos\alpha$$

With these two equations, the values of $k_{x+\delta x}$ and $A_{x+\delta x}$ can be calculated. By knowing the value of $k_{x+\delta x}$ the value of $G_{x+\delta x}$can be calculated.

$$G_{x+\delta x} = e^{\ln G_x - \frac{1}{2}(\ln k_{x+\delta x} - \ln k_x)}$$

Wave direction at point $x + \delta x$, calculated by Equation,

$$\alpha = \operatorname{atan}\left(\frac{u}{v}\right)$$

$$\frac{\partial\alpha}{\partial x} = \frac{u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x}}{u^2 + v^2}$$

In the following section, we present the results of the refraction-diffraction model applied to submerged island bathymetry (Fig. 3). The model considers a wave with a period $T = 8.0$ sec and amplitude $H = 2.4$ m, incident at an $\alpha = 0^0$relative to the horizontal $-x$. The calculation parameters used are $\theta = 1.182$ and $\varepsilon = 0.01$.

Figures 4 and 5 depict the results of the refraction-diffraction model. These figures demonstrate that the model effectively simulates both refraction-diffraction and shoaling-breaking phenomena.

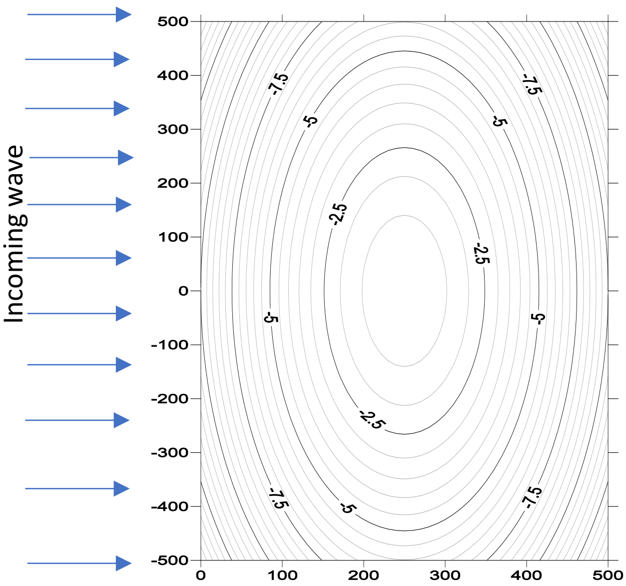

Fig (3) Submerged island bathymetric contour.

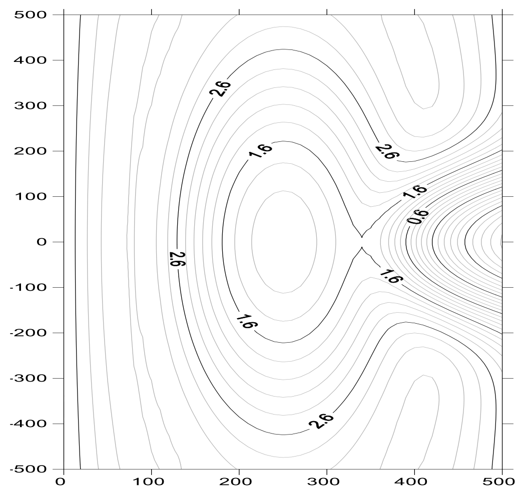

Fig (4) Wave height contour on the submerged island

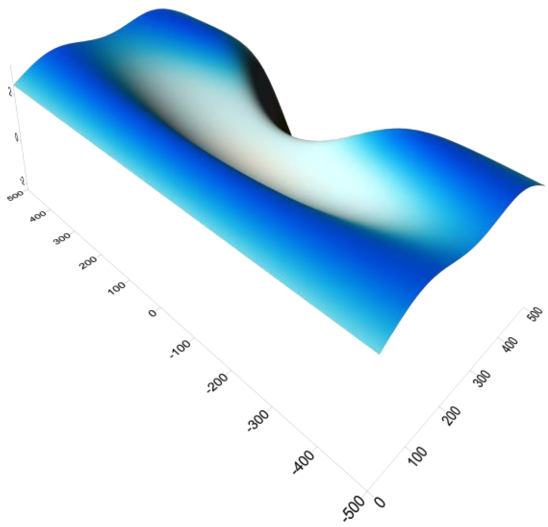

Fig (5) 3-D Wave height, on submerged island

IX. CONCLUSION

Increasing the accuracy of KFSBC integration leads to improvements in both critical wave steepness and breaking wave steepness. Moving from zero-order integration accuracy to second-order accuracy results in significant changes in both critical wave steepness and breaking wave steepness. However, the transition from second-order to third-order accuracy shows minimal changes in these parameters, indicating convergence of integration.

The first conclusion drawn is that the integration method employed is correct and suitable for the analysis. Furthermore, the convergence of integration results suggests that third-order accuracy can be considered sufficient, with the resulting wave amplitude function showing excellent performance.

The critical wave steepness and breaking wave steepness obtained represent final values and can serve as reliable references. Similarly, the resulting breaking wave height is accurate.

Although analytical equations for calculating breaker depth or breaker depth index were not derived in this research, the shoaling model provides estimates of breaker depth. There exists a correlation between deep water depth and breaker depth: greater deep water depth corresponds to greater breaker depth, and vice versa. Certainty about breaker depth implies certainty about deep water depth, and vice versa. Currently, the breaker depth index derived from empirical research by previous investigators serves as an approach for estimating deep water depth based on breaker depth.

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